

A fractional Yamabe flow and some applications

Tianling Jin and Jingang Xiong

October 27, 2011

Abstract

We introduce a fractional Yamabe flow involving nonlocal conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds, and show that on the conformal spheres $(\mathbb{S}^n, [g_{\mathbb{S}^n}])$, it converges to the standard sphere up to a Möbius diffeomorphism. This result allows us to obtain extinction profiles of solutions of some fractional porous medium equations. In the end, we use this fractional fast diffusion equation, together with its extinction profile and some estimates of its extinction time, to improve a Sobolev inequality via a quantitative estimate of the remainder term.

1 Introduction

Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 2$. The following evolution equation for the metric g

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - r_{g(t)})g(t), \quad g(0) = g_0 \quad (1)$$

was introduced by Hamilton in [41], and is known as the Yamabe flow. Here, $R_{g(t)}$ is the scalar curvature of $g(t)$ and $r_{g(t)} = \text{vol}_{g(t)}(M)^{-1} \int_M R_{g(t)} d\text{vol}_g$ is the average of $R_{g(t)}$. The existence and convergence of (1) were established through [41], [24], [72], [66], [12] and [14].

In [38], Graham, Jenne, Mason and Sparling constructed a sequence of conformally invariant elliptic operators, $\{P_k^g\}$, on (M, g) for all positive integer k if n is odd, and for $k \in \{1, \dots, n/2\}$ if n is even. Moreover, P_1^g is the conformal Laplacian $-L_g := -\Delta_g + c(n)R_g$ and P_2^g is the Paneitz operator. Up to a positive constant $P_1^g(1)$ is the scalar curvature of g and $P_2^g(1)$ is the Q -curvature. Some higher integer order curvature flows involving P_k^g , such as Q -curvature flow, have been studied in [11], [57], [4], [13], [44], etc.

Making use of a generalized Dirichlet to Neumann map, Graham and Zworski [39] introduced a meromorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds. Recently, Chang and González [21] reconciled the way of Graham and Zworski to define conformally invariant operators P_σ^g of non-integer order

$\sigma \in (0, \frac{n}{2})$ and the localization method of Caffarelli and Silvestre [18] for fractional Laplacian $(-\Delta)^\sigma$ on the Euclidean space \mathbb{R}^n . These lead naturally to a fractional order curvature $R_\sigma^g := P_\sigma^g(1)$, which will be called σ -curvature in this paper. There have been several work on these conformally invariant equations of fractional order and prescribing σ -curvatures problem (fractional Yamabe problem and fractional Nirenberg problem), see, e.g., [64], [36], [37], [48] and [49]. In this paper we study some flow of this fractional order curvature R_σ^g associated with P_σ^g on the standard conformal sphere $(\mathbb{S}^n, [g_{\mathbb{S}^n}])$, which is the conformal infinity of the Poincaré disk with the standard Poincaré metric.

Let g be a representative in the conformal class $[g_{\mathbb{S}^n}]$ and write $g = v^{\frac{4}{n-2\sigma}} g_{\mathbb{S}^n}$, where v is positive and smooth on \mathbb{S}^n . Then we have

$$P_\sigma^g(\phi) = v^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma^{g_{\mathbb{S}^n}}(\phi v) \quad \text{for any } \phi \in C^\infty(\mathbb{S}^n). \quad (2)$$

$P_\sigma^{g_{\mathbb{S}^n}}$, which is simply written as P_σ , has the formula (see, e.g., [9] or [61])

$$P_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2}, \quad (3)$$

where Γ is the Gamma function and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace-Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. Let $Y^{(k)}$ be a spherical harmonic of degree $k \geq 0$. Since $-\Delta_{g_{\mathbb{S}^n}} Y^{(k)} = k(k+n-1)Y^{(k)}$,

$$B(Y^{(k)}) = \left(k + \frac{n-1}{2}\right) Y^{(k)} \quad \text{and} \quad P_\sigma(Y^{(k)}) = \frac{\Gamma(k + \frac{n}{2} + \sigma)}{\Gamma(k + \frac{n}{2} - \sigma)} Y^{(k)}. \quad (4)$$

It is also well-known (see, e.g., [61]) that P_σ is the inverse of the spherical Riesz potential

$$K^\sigma(f)(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{f(\zeta)}{|\xi - \zeta|^{n-2\sigma}} d\text{vol}_{g_{\mathbb{S}^n}}(\zeta), \quad f \in L^p(\mathbb{S}^n) \quad (5)$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma} \pi^{n/2} \Gamma(\sigma)}$, $1 \leq p < \infty$ and $|\cdot|$ is the Euclidean distance in \mathbb{R}^{n+1} . The inverses of spherical Riesz potentials have been constructed in terms of singular integrals in [63] and [65]. When $\sigma \in (0, 1)$, Pavlov and Samko [63] showed that if $v = K^\sigma(f)$ for some $f \in L^p(\mathbb{S}^n)$, then

$$P_\sigma(v)(\xi) = P_\sigma(1)v(\xi) + c_{n,-\sigma} \int_{\mathbb{S}^n} \frac{v(\xi) - v(\zeta)}{|\xi - \zeta|^{n+2\sigma}} d\text{vol}_{g_{\mathbb{S}^n}}(\zeta), \quad (6)$$

where $c_{n,-\sigma} = \frac{2^{2\sigma} \sigma \Gamma(\frac{n+2\sigma}{2})}{\pi^{\frac{n}{2}} \Gamma(1-\sigma)}$ and $\int_{\mathbb{S}^n}$ is understood as $\lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon}$ in $L^p(\mathbb{S}^n)$ sense.

Consider the normalized total σ -curvature functional

$$S(g) = \text{vol}_g(\mathbb{S}^n)^{\frac{2\sigma-n}{n}} \int_{\mathbb{S}^n} R_\sigma^g d\text{vol}_g, \quad g \in [g_{\mathbb{S}^n}].$$

The negative gradient flow of S takes the form

$$\frac{\partial g}{\partial t} = -\frac{n-2\sigma}{2n}(\text{vol}_g(\mathbb{S}^n))^{\frac{2\sigma-n}{n}}(R_\sigma^g - r_\sigma^g)g$$

where r_σ^g is the average of R_σ^g . It is easy to verify that this flow preserves the conformal class and the volume of \mathbb{S}^n . By a rescaling of the time variable, we obtain the following evolution equation

$$\frac{\partial g}{\partial t} = -(R_\sigma^g - r_\sigma^g)g. \quad (7)$$

If we write $g(t) = v^{\frac{4}{n-2\sigma}}(\cdot, t)g_{\mathbb{S}^n}$, then after rescaling the time variable, (7) can be written in an equivalent form

$$\frac{\partial v^N}{\partial t} = -P_\sigma(v) + r_\sigma^g v^N, \quad \text{on } \mathbb{S}^n, \quad (8)$$

where $N = (n+2\sigma)/(n-2\sigma)$.

Let \mathcal{N} be the north pole of \mathbb{S}^n and

$$F : \mathbb{R}^n \rightarrow \mathbb{S}^n \setminus \{\mathcal{N}\}, \quad x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1} \right)$$

be the inverse of stereographic projection from $\mathbb{S}^n \setminus \{\mathcal{N}\}$ to \mathbb{R}^n . Then

$$(P_\sigma(\phi)) \circ F = |J_F|^{-\frac{n+2\sigma}{2n}}(-\Delta)^\sigma(|J_F|^{\frac{n-2\sigma}{2n}}(\phi \circ F)), \quad \text{for } \phi \in C^\infty(\mathbb{S}^n) \quad (9)$$

where $|J_F| = \left(\frac{2}{1+|x|^2} \right)^n$ and $(-\Delta)^\sigma$ is the fractional Laplacian operator (see, e.g., [69]).

Hence $u(x, t) := |J_F|^{\frac{n-2\sigma}{2n}}v(F(x), t)$ satisfies

$$\frac{\partial u^N}{\partial t} = -(-\Delta)^\sigma u + r_\sigma^g u^N, \quad \text{in } \mathbb{R}^n. \quad (10)$$

We will call (7), (8) or (10) as a (normalized) *fractional Yamabe flow* when $\sigma \in (0, 1)$.

As observed in [21] that the operator $P_{1/2}^g$ is related to the Yamabe problem on manifolds with boundary (see, e.g., [23, 32, 33, 42, 43]), this fractional Yamabe flow (7) with $\sigma = 1/2$ is related to some generalization of Yamabe flow for manifolds with boundary studied in [10].

Throughout this paper we always assume that $0 < \sigma < 1$ without otherwise stated. Our first result is the long time existence and convergence of solutions of (7) for any initial data in the conformal class of $g_{\mathbb{S}^n}$.

Theorem 1.1. *Let $g(0) \in [g_{\mathbb{S}^n}]$ be a smooth metric on \mathbb{S}^n . Then the fractional Yamabe flow (7) with initial metric $g(0)$ exists for all time $0 < t < \infty$. Furthermore, there exist a smooth metric $g_\infty \in [g_{\mathbb{S}^n}]$ such that*

$$R_\sigma^{g_\infty} = r_\sigma^{g_\infty} \quad \text{and} \quad \lim_{t \rightarrow \infty} \|g(t) - g_\infty\|_{C^l} = 0$$

for all positive integers l .

Remark 1.1. If we write $g_\infty = v_\infty^{\frac{4}{n-2\sigma}} g_{\mathbb{S}^n}$ where v_∞ is a smooth and positive function on \mathbb{S}^n , then Theorem 1.1 tells that v_∞ satisfies

$$P_\sigma(v_\infty) = r_\sigma^{g_\infty} \cdot v_\infty^{\frac{n+2\sigma}{n-2\sigma}},$$

whose solutions are classified in [22] and [54].

We also consider the unnormalized fractional Yamabe flow

$$\frac{\partial v^N}{\partial t} = -P_\sigma(v) \quad \text{on } \mathbb{S}^n \times (0, \infty), \quad \text{or} \quad \frac{\partial u^N}{\partial t} = -(-\Delta)^\sigma u \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

The second one is a fractional porous medium equation studied in [2], [27], [19], [28] and [50], where it is taken the form

$$\begin{cases} u_t = -(-\Delta)^\sigma(|u|^{m-1}u) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \quad (11)$$

with $m = \frac{n-2\sigma}{n+2\sigma}$, $\sigma \in (0, 1)$. Models of this kind of fractional diffusion equations arise, e.g., in statistical mechanics [45, 46, 47] and heat control [2].

These fractional diffusion equations have been systematically studied in [27] and [28]. It is proved in [28] that if $u_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $p > 4n/(n+2\sigma)$, then there exists a unique strong solution (see [28] for the definition) of (11), and the solution will extinct in finite time. More precisely, if u_0 is nonnegative but not identically equals to zero, then there exists a $T = T(u_0) \in (0, \infty)$ such that $u(x, t) > 0$ in $\mathbb{R}^n \times (0, T)$, and $u(x, T) \equiv 0$ in \mathbb{R}^n .

We are interested in analyzing the exact behavior of solutions of (11) near the extinction time for fast decaying initial data. In the classical case $\sigma = 1$, the extinction profiles of solutions of porous medium equations have been described in the results of [34, 29, 26, 7, 8] and so on.

Theorem 1.2 below describes the extinction profile of $u(x, t)$, which extends the result of del Pino and Saéz in [29] to $\sigma \in (0, 1)$. Some estimates of the extinction time T involving the sharp constant in the Hardy-Littlewood-Sobolev inequality are postponed to Lemma 6.1 in Section 6.

Theorem 1.2. Assume that $u_0(x) \in C^2(\mathbb{R}^n)$ is positive in \mathbb{R}^n . In addition, we assume, for $(u_0^m)_{0,1}(x) := |x|^{2\sigma-n} u_0^m(x/|x|^2)$, that $(u_0^m)_{0,1}(x)$ can be extended to a positive and C^2 function near the origin. There exist $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that if $T = T(u_0) \in (0, \infty)$ denotes the extinction time of the solution of (11), then

$$(T - t)^{-1/(1-m)} u(x, t) = k(n, \sigma) \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n+2\sigma}{2}} + \theta(x, t)$$

with

$$\sup_{\mathbb{R}^n} (1 + |x|^{n+2\sigma}) \theta(x, t) \rightarrow 0 \quad \text{as } t \rightarrow T,$$

where $k(n, \sigma) = 2^{\frac{n-2}{2}} ((1-m)P_\sigma(1))^{\frac{n-2\sigma}{4\sigma}}$ and $P_\sigma(1)$ is given in (4).

An application of Theorem 1.2 is an improvement of some Sobolev inequality. A sharp form of the standard Sobolev inequality in \mathbb{R}^n ($n \geq 3$) asserts that

$$S_n \|\nabla u\|_{L^2(\mathbb{R}^n)} - \|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \geq 0 \quad (12)$$

for all $u \in \dot{H}^1(\mathbb{R}^n) = \{u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\}$, where S_n is the sharp constant obtained in [3] and [70].

There have seen many results on remainder terms of Sobolev inequalities (see, e.g., [16, 15, 6, 25, 20, 31]), which give various lower bounds of the left-handed side of (12).

Recently, Carlen, Carrillo and Loss in [20] noticed that some Hardy-Littlewood-Sobolev inequalities in dimension $n \geq 3$ and some special Gagliardo-Nirenberg inequalities can be related by a fast diffusion equation. In another recent paper [31], Dolbeault used a fast diffusion flow to obtain an optimal integral remainder term which improves (12) in dimension $n \geq 5$. Inspired by [20] and [31], we consider some Sobolev inequality involving fractional Sobolev spaces of order $\sigma \in (0, 1)$, compared to those mentioned above corresponding to $\sigma = 1$.

For any $\sigma \in (0, 1)$, the Sobolev inequality (see, e.g., [69] or [30]) asserts that

$$\|u\|_{L^{2^*(\sigma)}}^2 \leq S_{n,\sigma} \|u\|_{\dot{H}^\sigma}^2, \quad \forall u \in \dot{H}^\sigma(\mathbb{R}^n) \quad (13)$$

where $2^*(\sigma) = \frac{2n}{n-2\sigma}$, $S_{n,\sigma}$ is the optimal constant and $\dot{H}^\sigma(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{\dot{H}^\sigma} = \|(-\Delta)^{\sigma/2} u\|_{L^2(\mathbb{R}^n)}. \quad (14)$$

The optimal constant $S_{n,\sigma}$ in the Sobolev inequality (13) is attained by (see, e.g., [55], [22] and [54]) $u(x) = (1 + |x|^2)^{-\frac{n-2\sigma}{2}}$. The Hardy-Littlewood-Sobolev inequality

$$S_{n,\sigma} \|u\|_{L^{\frac{2n}{n+2\sigma}}}^2 \geq \int_{\mathbb{R}^n} u(-\Delta)^{-\sigma} u \, dx, \quad \forall u \in L^{\frac{2n}{n+2\sigma}}(\mathbb{R}^n) \quad (15)$$

involves the same optimal constant $S_{n,\sigma}$, where $(-\Delta)^{-\sigma}$ is a Riesz potential defined by

$$(-\Delta)^{-\sigma} u(x) = c_{n,\sigma} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-2\sigma}} dy. \quad (16)$$

We investigate the relation between (13) and (15) via the fractional diffusion equation (11), i.e.

$$u_t = -(-\Delta)^\sigma u^m$$

with $m = 1/N = \frac{n-2\sigma}{n+2\sigma}$. If we suppose that the initial data u_0 satisfies the assumptions in Theorem 1.2, then by Theorem 5.1 (which is used to prove Theorem 1.2) $u(\cdot, t)$ is positive and smooth in \mathbb{R}^n before its extinction time, and for any fixed t , $u(x, t) = O(|x|^{-n-2\sigma})$ as $x \rightarrow \infty$. We define

$$H(t) := H_{n,\sigma}(u(\cdot, t)) = \int_{\mathbb{R}^n} u(-\Delta)^{-\sigma} u dx - S_{n,\sigma} \|u\|_{L^{\frac{2n}{n+2\sigma}}}^2. \quad (17)$$

It follows from direct computations that $\frac{d}{dt}H \geq 0$ (see Proposition 6.1).

Consequently, one can prove (15), which is equivalent to $H \leq 0$, by showing

$$\limsup_{t \rightarrow T} H(t) \leq 0$$

where T is the extinction time of (11). This can be seen clearly from Theorem 5.1. From this and Proposition 6.1 we also recover that u^m is an extremal of (13) if u is an extremal of (15).

Along this fractional fast diffusion flow, we can improve the Sobolev inequality (13), via a quantitative estimate of the remainder term. This improvement also holds as $\sigma \rightarrow 1$ and it extends some work of Dolbeault in [31].

Theorem 1.3. *Assume that $\sigma \in (0, 1)$ and $n > 4\sigma$. There exists a positive constant C depending only on n and σ such that for any nonnegative function $u \in \dot{H}^\sigma(\mathbb{R}^n)$ we have*

$$\begin{aligned} S_{n,\sigma} \|u^N\|_{L^{\frac{2n}{n+2\sigma}}}^2 - \int_{\mathbb{R}^n} u^N (-\Delta)^{-\sigma} u^N dx \\ \leq C \|u\|_{L^{2^*(\sigma)}^{\frac{8\sigma}{n-2\sigma}}} \left(S_{n,\sigma} \|u\|_{\dot{H}^\sigma}^2 - \|u\|_{L^{2^*(\sigma)}}^2 \right), \end{aligned} \quad (18)$$

where $N = \frac{n+2\sigma}{n-2\sigma}$. Moreover $C \leq \frac{n+2\sigma}{n} (1 - e^{-\frac{n}{2\sigma}}) S_{n,\sigma}$.

The operators P_σ and $(-\Delta)^\sigma$ are nonlocal, pseudo-differential operators. Generally speaking, strong maximum principles and Harnack inequalities might fail for nonlocal operators, see, e.g., a counterexample in [51]. The example in [51] shows that the local non-negativity of solutions of certain nonlocal equations is not enough to obtain local strong maximum principles and Harnack inequalities. However, if we assume the solutions are globally nonnegative, then various strong maximum principles and Harnack inequalities have been obtained in, e.g., [17], [71] and [48].

In this paper, we establish a strong maximum principle and a Hopf lemma for odd solutions of some linear nonlocal parabolic equations, which are of independent interest. Our proofs make full use of the expression (19) of $(-\Delta)^\sigma$. The odd function in Lemma 2.4 will serve as a barrier function, which allows us to obtain a Hopf lemma.

The paper is organized as follows. In Section 2 we prove a strong maximum principle and a Hopf lemma for odd solutions of some linear nonlocal parabolic equations. These are two crucial ingredients in our arguments. In Section 3 we prove a Harnack inequality via

the method of moving planes. This idea is essentially due to R. Ye [72]. In Section 4, we show Schauder estimates, existence and convergence of the fractional Yamabe flows. Section 5 is devoted to proving Theorem 1.2. The strategy is to rewrite (11) on \mathbb{S}^n and apply the methods in Section 4. The improvement of the Sobolev inequality, Theorem 1.3, is proved in Section 6. Our proofs of Theorem 1.2 and Theorem 1.3 adapt some arguments in [29] and [31], respectively. In Appendix A we provide an analog of L. Simon's uniqueness theorem (see [68]) for negative gradient flows in our nonlocal flow setting. In Appendix B we present some interpolation inequalities and elementary computations which are used in Section 4.

Acknowledgements: Both authors thank Prof. Y.Y. Li for encouragements and useful discussions. We also thank Prof. N. Sesum for bringing the paper [29]. Tianling Jin was partially supported by a University and Bevier Fellowship and Rutgers University School of Art and Science Excellence Fellowship. Jingang Xiong was partially supported by CSC project for visiting Rutgers University and NSFC No. 11071020. He is very grateful to the Department of Mathematics at Rutgers University for the kind hospitality.

2 A strong maximum principle and a Hopf lemma for nonlocal parabolic equations

Let $x = (x', x_n) \in \mathbb{R}^n$, $\mathbb{R}_+^n = \{x : x_n > 0, x \in \mathbb{R}^n\}$. Recall (see, e.g., [67]) that for $\sigma \in (0, 1)$, if u is bounded in \mathbb{R}^n and C^2 near x , then $(-\Delta)^\sigma u$ is continuous near x , and

$$(-\Delta)^\sigma u(x) = c_{n,-\sigma} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy. \quad (19)$$

Here “P.V.” means the principal value and $c_{n,-\sigma}$ is the constant in (6).

For simplicity, throughout the paper we denote $-(-\Delta)^\sigma$ as Δ^σ and will not keep writing the constant $c_{n,-\sigma}$ and “P.V.” if there is no confusion.

Lemma 2.1. *Let $w(x, t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R})$ and $w(\cdot, t)$ be bounded in \mathbb{R}^n for any fixed t . Suppose $w(x, t)$ satisfies $w(x', -x_n, t) = -w(x', x_n, t)$ for all (x, t) and*

$$\liminf_{x_n \geq 0, |x| \rightarrow \infty} w(x, t) \geq 0 \quad \text{for any fixed } t.$$

If w satisfies

$$w_t \geq a(x, t) \Delta^\sigma w + b(x, t) w, \quad (x, t) \in \mathbb{R}_+^n \times (0, T] \quad (20)$$

where $a(x, t)$ is continuous and positive in $\overline{\mathbb{R}_+^n} \times [0, T]$, $b(x, t)$ is continuous and bounded in $\mathbb{R}_+^n \times [0, T]$, and $w(x, 0) \geq 0$ for all $x \in \mathbb{R}_+^n$, then $w(x, t) \geq 0$ in $\mathbb{R}_+^n \times [0, T]$.

Proof. Without loss of generality, we may assume $b(x, t) \leq 0$. Indeed, if we let

$$\tilde{w}(x, t) = e^{-Ct}w(x, t),$$

then

$$\tilde{w}_t = a(x, t)\Delta^\sigma \tilde{w} + (b(x, t) - C)\tilde{w}.$$

For large constant C we have $b(x, t) - C \leq 0$ in $\mathbb{R}_+^n \times (0, T]$ and we only need to show that $\tilde{w}(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}_+^n \times (0, T]$.

Suppose the contrary that there exists a point $(x_0, t_0) \in \mathbb{R}_+^n \times (0, T]$ such that

$$0 > w(x_0, t_0).$$

By the assumptions on w , we may assume $w(x_0, t_0) = \min_{\overline{\mathbb{R}_+^n} \times (0, T]} w$. It follows that

$$w_t(x_0, t_0) \leq 0, \quad b(x_0, t_0)w(x_0, t_0) \geq 0. \quad (21)$$

It is clear that

$$\begin{aligned} \Delta^\sigma w(x_0, t_0) &= \int_{\mathbb{R}^n} \frac{w(y, t_0) - w(x_0, t_0)}{|x_0 - y|^{n+2\sigma}} dy \\ &= \int_{\mathbb{R}_+^n} \frac{w(y, t_0) - w(x_0, t_0)}{|x_0 - y|^{n+2\sigma}} dy + \int_{\mathbb{R}^n \setminus \mathbb{R}_+^n} \frac{w(y, t_0) - w(x_0, t_0)}{|x_0 - y|^{n+2\sigma}} dy. \end{aligned}$$

By the change of variables $y = (z', -z_n)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \mathbb{R}_+^n} \frac{w(y, t_0) - w(x_0, t_0)}{|x_0 - y|^{n+2\sigma}} dy \\ &= \int_{\mathbb{R}_+^n} \frac{w(z', -z_n, t_0) - w(x_0, t_0)}{|x_0 - (z', -z_n)|^{n+2\sigma}} dz \\ &= - \int_{\mathbb{R}_+^n} \frac{w(z', z_n, t_0) - w(x_0, t_0)}{|x_0 - (z', -z_n)|^{n+2\sigma}} dz - 2w(x_0, t_0) \int_{\mathbb{R}_+^n} \frac{1}{|x_0 - (z', -z_n)|^{n+2\sigma}} dz \\ &> - \int_{\mathbb{R}_+^n} \frac{w(z', z_n, t_0) - w(x_0, t_0)}{|x_0 - (z', -z_n)|^{n+2\sigma}} dz. \end{aligned}$$

where we used $w(z', -z_n, t_0) = -w(z', z_n, t_0)$ and $w(x_0, t_0) < 0$. Since (x_0, t_0) is a minimum point of w in $\overline{\mathbb{R}_+^n} \times (0, T]$, the simple inequality

$$\frac{1}{|x_0 - z|^{n+2\sigma}} > \frac{1}{|x_0 - (z', -z_n)|^{n+2\sigma}}, \quad \forall x_0, z \in \mathbb{R}_+^n$$

yields that

$$\Delta^\sigma w(x_0, t_0) > 0. \quad (22)$$

Combining (21) and (22), we have a contradiction to (20). \square

Lemma 2.2. *Let $w(x, t)$ be as in Lemma 2.1. Then for any fixed $t \in (0, T]$ we have $w(x, t) > 0$ or $w(x, t) \equiv 0$ in \mathbb{R}_+^n .*

Proof. As in the proof of Lemma 2.1, we may assume $b \leq 0$. Suppose that at $w(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathbb{R}_+^n \times (0, T]$. From the proof of Lemma 2.1 we see that

$$\Delta^\sigma w(x_0, t_0) \geq 0$$

and equality holds if and only if $w(x, t_0) = w(x_0, t_0)$ for all $x \in \mathbb{R}_+^n$. Therefore, the lemma follows immediately from a simple contradiction argument. \square

Lemma 2.3. *Let $w(x, t)$ be as in Lemma 2.1. Suppose $w(x_0, 0) > 0$ for some $x_0 \in \mathbb{R}_+^n$, then for any fixed $t \in (0, T]$ we have $w(x, t) > 0$ in \mathbb{R}_+^n .*

Proof. The proof follows from that of the parabolic strong maximum principle in [62], with modifications for nonlocal equations. As before we assume $b \leq 0$. Suppose that for some $t_1 > 0$, $w(\cdot, t_1)$ is zero at some point. It follows from Lemma 2.2 that $w(x, t_1) \equiv 0$. By the assumption on $w(\cdot, 0)$ and Lemma 2.2, we may assume that $w(x, t) > 0$ for all $(x, t) \in \mathbb{R}_+^n \times (t_2, t_1)$ for some $t_2 > 0$.

Let $h(x, t) = (t_1 - t_*)^2 - |x - e_n|^2 - (t - t_*)^2$ if $0 \leq x_n \leq 2$; and $h = 0$ if $x_n > 2$, where $e_n = (0', 1)$ and t_* will be fixed later. Set

$$H(x, t) = \begin{cases} h(x, t), & x \in \overline{\mathbb{R}_+^n}, \\ -h(x', -x_n, t), & x \in \mathbb{R}^n \setminus \overline{\mathbb{R}_+^n}. \end{cases}$$

Let $\bar{t} \in (t_2, t_1)$ be such that $(t_1 - t_*)^2 - (t - t_*)^2 \leq \frac{1}{4}$ holds for $t \geq \bar{t}$. It is easy to see that there exists a positive constant C_1 independent of t^* such that for any $(x, t) \in B_{1/2}(e_n) \times [\bar{t}, t_1]$,

$$(-\Delta)^\sigma H(x, t) \leq C_1.$$

Thus we can choose t_* so negative that for any $(x, t) \in B_{1/2}(e_n) \times [\bar{t}, t_1]$,

$$H_t(x, t) = 2(t_* - t) \leq 2(t_* - t_2) < a(x, t)\Delta^\sigma H(x, t) + b(x, t)H(x, t). \quad (23)$$

Let $\varepsilon > 0$ be a sufficiently small constant such that $w(x, \bar{t}) \geq \varepsilon H(x, \bar{t})$ for all $x \in \mathbb{R}_+^n$. We claim that $w(x, t) \geq \varepsilon H(x, t)$ in $\mathbb{R}_+^n \times (\bar{t}, t_1)$.

If not, then the (negative) minimal value of $\bar{w} := w - \varepsilon H$ in $\mathbb{R}_+^n \times (\bar{t}, t_1)$ must be achieved in $B_{1/2}(e_n) \times (\bar{t}, t_1)$, say at (x_0, t_0) . Note that $\bar{w}(x', -x_n, t) = -\bar{w}(x', x_n, t)$. Hence, by exactly the same argument in the proof of Lemma 2.1

$$\partial_t \bar{w}(x_0, t_0) \leq 0, \quad \Delta^\sigma \bar{w}(x_0, t_0) > 0.$$

Together with (23) and $b \leq 0$, we conclude that

$$w_t(x_0, t_0) < a(x_0, t_0)\Delta^\sigma w(x_0, t_0) + b(x_0, t_0)w,$$

which contradicts (20).

Hence $w_t(t_1, e_n) \leq -2\varepsilon(t_1 - t_*) < 0$. But $w(x, t_1) \equiv 0$. These contradict (20). \square

Lemma 2.4. *Let*

$$h(x) = \begin{cases} x_n(1 - |x'|^2), & |x_n| < 1, |x'| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists a positive constant c_0 depending only n, σ such that

$$\Delta^\sigma h(x) \geq -c_0 x_n, \quad (24)$$

for all $x = (x', x_n)$ with $|x'| < 1, 0 \leq x_n < 1/8$.

Proof. The lemma follows from rather involved calculations. By rotating the first $(n-1)$ coordinates, we only need to show (24) at point $a = (a_1, 0, \dots, 0, a_n)$ with $0 \leq a_1 < 1, 0 \leq a_n < 1/8$.

Denote $B'(x', R) \subset \mathbb{R}^{n-1}$ be the ball centered at x' with radius R , $\Omega = B'(0, 1) \times (-1, 1)$. In the following C will be denoted as various positive constants which depend only on dimension n and σ .

It follows from (19) that

$$\begin{aligned} \Delta^\sigma h(a) &= \int_{\mathbb{R}^n} \frac{h(x) - h(a)}{|x - a|^{n+2\sigma}} dx \\ &= \int_{\Omega} \frac{x_n(1 - |x'|^2) - a_n(1 - |a'|^2)}{|x - a|^{n+2\sigma}} dx - \int_{\Omega^c} \frac{a_n(1 - |a'|^2)}{|x - a|^{n+2\sigma}} dx \\ &=: I - a_n II. \end{aligned} \quad (25)$$

Since $x_n(1 - |x'|^2) - a_n(1 - |a'|^2) = (x_n - a_n)(1 - |x'|^2) + a_n(|a'|^2 - |x'|^2)$, we divide the integral I into

$$I_1 := \int_{\Omega} \frac{(x_n - a_n)(1 - |x'|^2)}{|x - a|^{n+2\sigma}} dx$$

and

$$a_n I_2 := a_n \int_{\Omega} \frac{(|a'|^2 - |x'|^2)}{|x - a|^{n+2\sigma}} dx.$$

By symmetry and that $0 \leq a_n < 1/8$,

$$\begin{aligned} I_1 &= \int_{-1}^{-1+2a_n} \int_{|x'| < 1} \frac{(x_n - a_n)(1 - |x'|^2)}{|x - a|^{n+2\sigma}} dx' dx_n \\ &\geq -C a_n. \end{aligned} \quad (26)$$

Using $|a'|^2 - |x'|^2 = -|x' - a'|^2 + 2a' \cdot (a' - x')$, we write

$$\begin{aligned} I_2 &= \int_{\Omega} \frac{-|x' - a'|^2}{|x - a|^{n+2\sigma}} dx + \int_{\Omega} \frac{2a' \cdot (a' - x')}{|x - a|^{n+2\sigma}} dx \\ &=: I_3 + I_4. \end{aligned}$$

Direct computations give

$$\begin{aligned}
I_3 &\geq - \int_{-2+a_n}^{2+a_n} dx_n \int_{|x'-a'|<2} \frac{-|x'-a'|^2}{|x-a|^{n+2\sigma}} dx' \\
&= -2 \lim_{b \rightarrow 0^+} \int_b^2 dy \int_0^2 \frac{r^2 r^{n-2}}{(r^2 + y^2)^{\frac{n+2\sigma}{2}}} dr \\
&= -2 \lim_{b \rightarrow 0^+} \int_b^2 y^{1-2\sigma} dy \int_0^{2/y} \frac{r^n}{(1+r^2)^{\frac{n+2\sigma}{2}}} dr \\
&= -2 \lim_{b \rightarrow 0^+} \int_b^2 y^{1-2\sigma} dy \left(\int_1^{2/y} \frac{r^n}{(1+r^2)^{\frac{n+2\sigma}{2}}} dr + \int_0^1 \frac{r^n}{(1+r^2)^{\frac{n+2\sigma}{2}}} dr \right) \\
&\geq -2 \lim_{b \rightarrow 0^+} \int_b^2 y^{1-2\sigma} dy \left(\int_1^{2/y} r^{-2\sigma} dr + 1 \right) \\
&\geq -C.
\end{aligned} \tag{27}$$

Next, we are going to show

$$I_4 - II \geq -C. \tag{28}$$

Let $D_0 = (B'(0, 1) \cap B'(2a', 1))$. Since $a' = (a_1, 0, \dots, 0)$, it follows from symmetry that

$$\int_{D_0 \times (-1, 1)} \frac{2a' \cdot (a' - x')}{|x - a|^{n+2\sigma}} dx' dx_n = 0.$$

Thus

$$I_4 = \int_{(B'(0, 1) \setminus D_0) \times (-1, 1)} \frac{2a' \cdot (a' - x')}{|x - a|^{n+2\sigma}} > 0.$$

Now we have two cases:

Case 1. if $|a'| \leq \frac{\sqrt{2}}{2}$, then it is easy to see that $II < C$ (the denominator is uniformly bounded). Hence (28) holds.

Case 2. Suppose $|a'| > \frac{\sqrt{2}}{2}$. We have

$$\begin{aligned}
II &= \int_{\Omega^c \cap (B'(a', |a'|) \times (-1, 1))} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}} + \int_{\Omega^c \setminus (B'(a', |a'|) \times (-1, 1))} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}} \\
&\leq \int_{\Omega^c \cap (B'(a', |a'|) \times (-1, 1))} \frac{(1 - |a'|^2)}{|x - a|^{n+2\sigma}} + C \\
&=: II_1 + C.
\end{aligned}$$

Denote $D_1 = \left(B'(a', \sqrt{1 - |a'|^2}) \cap \{x_1 < a_1\} \right) \setminus D_0$, and $D_2 = \left(B'(a', \sqrt{1 - |a'|^2}) \cap \{x_1 > a_1\} \right) \setminus D_0$.

Note that for $x \in D_1$, we have $2|a'|(|a'| - x_1) \geq 1 - |a'|^2 - |x' - a'|^2$. Therefore,

$$\int_{D_1 \times (-1,1)} \frac{2a' \cdot (a' - x')}{|x - a|^{n+2\sigma}} - \int_{D_2 \times (-1,1)} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}} \geq \int_{D_1 \times (-1,1)} \frac{-|x' - a'|^2}{|x - a|^{n+2\sigma}}.$$

Observe that there exists a positive integer m , which depends only on n and σ , such that

$$\begin{aligned} & m \int_{\left(B'(0',1) \setminus B'(a', \sqrt{1-|a'|^2})\right) \times (-1,1)} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}} \\ & \geq \int_{\left(B(a', |a'|) \setminus \left(B'(0',1) \cup B'(a', \sqrt{1-|a'|^2})\right)\right) \times (-1,1)} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}}. \end{aligned}$$

Also notice that for any $x \in B'(0', 1) \setminus B'(a', \sqrt{1 - |a'|^2})$, we have

$$0 \geq m(1 - |a'|^2 - |x' - a'|^2).$$

Hence,

$$\begin{aligned} & m \int_{\left(B'(0',1) \setminus B'(a', \sqrt{1-|a'|^2})\right) \times (-1,1)} \frac{|x' - a'|^2}{|x - a|^{n+2\sigma}} \\ & \geq \int_{\left(B(a', |a'|) \setminus \left(B'(0',1) \cup B'(a', \sqrt{1-|a'|^2})\right)\right) \times (-1,1)} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}}. \end{aligned}$$

It follows that

$$\begin{aligned} & I_4 - II \\ & \geq -C + \int_{D_1 \times (-1,1)} \frac{2a' \cdot (a' - x')}{|x - a|^{n+2\sigma}} - \int_{D_2 \times (-1,1)} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}} \\ & \quad - \int_{\left(B(a', |a'|) \setminus \left(B'(0',1) \cup B'(a', \sqrt{1-|a'|^2})\right)\right) \times (-1,1)} \frac{1 - |a'|^2}{|x - a|^{n+2\sigma}} \\ & \geq -C - m \int_{\left(B'(0',1) \setminus B'(a', \sqrt{1-|a'|^2})\right) \times (-1,1)} \frac{|x' - a'|^2}{|x - a|^{n+2\sigma}} + \int_{D_1 \times (-1,1)} \frac{-|x' - a'|^2}{|x - a|^{n+2\sigma}} \\ & \geq -C - (m+1)I_3 \\ & \geq -C. \end{aligned}$$

Therefore, (28) holds.

Finally, Lemma 2.4 follows from (25), (26), (27), and (28). \square

Lemma 2.5. *Let $w(x, t)$ be as in Lemma 2.1. Suppose $w(x_0, 0) > 0$ for some $x_0 \in \mathbb{R}_+^n$, then for any fixed $t \in (0, T]$ we have $\partial_{x_n} w(x', 0, t) > 0$.*

Proof. Let

$$g(x) = \begin{cases} -1, & \text{in } B'(0, 1) \times (-2, -1), \\ 1, & \text{in } B'(0, 1) \times (1, 2), \\ 0, & \text{otherwise,} \end{cases}$$

where $B'(0, 1)$ denotes the $n - 1$ dimensional unit ball centered at 0. For any $x \in B'(0, 1) \times (0, 1/8)$, we have

$$\begin{aligned} \Delta^\sigma g(x) &= \int_{\mathbb{R}^n} \frac{g(y) - g(x)}{|y - x|^{n+2\sigma}} dy \\ &= \int_{B'(0,1) \times (1,2)} \frac{1}{|y - x|^{n+2\sigma}} dy - \int_{B'(0,1) \times (-2,-1)} \frac{1}{|y - x|^{n+2\sigma}} dy \\ &= \int_{B'(0,1) \times (1,2)} \frac{1}{|y - (x', x_n)|^{n+2\sigma}} - \frac{1}{|y - (x', -x_n)|^{n+2\sigma}} dy \\ &= \int_{B'(0,1) \times (1,2)} \int_0^1 -\frac{d}{ds} \left(\frac{1}{|y - x + 2sx_n e_n|^{n+2\sigma}} \right) ds dy \\ &= (n + 2\sigma) \int_{B'(0,1) \times (1,2)} \int_0^1 \frac{4(y_n - x_n)x_n + 8sx_n^2}{|y - x + 2sx_n e_n|^{n+2+2\sigma}} ds dy \\ &\geq c_1 x_n, \end{aligned}$$

where $c_1 > 0$ depends only on n and σ .

For any fixed $t_0 \in (0, T]$, define

$$H(x, t) = h(x) \left(\frac{t_0^2}{1 + t_0^2} - (t - t_0)^2 \right) + kg(x),$$

where h is as in Lemma 2.4. By picking a sufficiently large k , we then see that

$$H_t(x, t) \leq a(x, t) \Delta^\sigma H + b(x, t) H(x, t),$$

for all $x \in B'(0, 1) \times (0, 1/8)$ and $t \in (t_0 - t_0/\sqrt{1 + t_0^2}, t_0]$.

It follows from Lemma 2.3 that $w(\cdot, t) > 0$ in \mathbb{R}_+^n for any fixed $t \in (0, T]$. Making a similar argument to the poof of Lemma 2.3, we can show that there exists a small positive constant ε such that $w \geq \varepsilon H$ for all $t \in (0, t_0]$. Therefore, $\partial_{x_n} w(x', 0, t_0) > 0$ and Lemma 2.5 follows immediately. \square

Now we apply the above strong maximum principle and Hopf lemma to fractional Yamabe flow equations.

Suppose that v is a positive smooth solution of (8) in $\mathbb{S}^n \times [0, T]$. Hence

$$u(x, t) = \left(\frac{2}{1 + |x|^2} \right)^{\frac{n-2\sigma}{2}} v(F(x), t)$$

satisfies (10). For a given real number λ , define

$$\Sigma_\lambda = \{x = (x', x_n) : x_n \geq \lambda\},$$

and let $x^\lambda = (x', 2\lambda - x_n)$ and $u_\lambda(x, t) = u(x^\lambda, t)$. It is clear that u_λ also satisfies (10).

Proposition 2.1. *Suppose that $u(x, 0) - u_\lambda(x, 0) \geq 0$ in Σ_λ , then for any fixed $t \in (0, T]$ we have $u(x, t) - u_\lambda(x, t) \geq 0$ in Σ_λ .*

Proof. Let $w(x, t) = u(x, t) - u_\lambda(x, t)$. Then w satisfies

$$w_t = a(x, t)\Delta^\sigma w + b(x, t)w \quad (29)$$

where $a(x, t) = \frac{1}{Nu^{N-1}}$ and $b(x, t) = \frac{(1-N)(-\Delta)^\sigma u_\lambda}{N} \int_0^1 \frac{1}{(\tau u + (1-\tau)u_\lambda)^N} d\tau + \frac{r_\sigma^g}{N}$ is bounded. Note that $w(x', x_n + \lambda, t)$ satisfies all the conditions in Lemma 2.1. Thus Proposition 2.1 follows from Lemma 2.1. \square

Proposition 2.2. *Assume the conditions in Proposition 2.1, then for any fixed $t \in (0, T]$ we have $u(x, t) - u_\lambda(x, t) > 0$ or $u(x, t) - u_\lambda(x, t) \equiv 0$ in Σ_λ .*

Proof. It follows from Proposition 2.1 and Lemma 2.2. \square

Proposition 2.3. *Assume the conditions in Proposition 2.1. In addition we suppose that $u(x_0, 0) - u_\lambda(x_0, 0) > 0$ for some $x_0 \in \Sigma_\lambda$, then for any fixed $t \in (0, T]$ we have $u(x, t) - u_\lambda(x, t) > 0$ in Σ_λ and $\partial_{x_n} u(x', \lambda, t) > 0$.*

Proof. It follows from Proposition 2.1, Lemma 2.3 and Lemma 2.5. \square

3 Harnack inequality for a fractional Yamabe flow

Based on the results proved in the previous section, we are going to establish the following Harnack inequality.

Theorem 3.1. *Let v be a $C^{3,1}$ positive function on $\mathbb{S}^n \times [0, T^*)$ and satisfy*

$$\frac{\partial v^N}{\partial t} = -P_\sigma(v) + b(t)v^N, \quad \text{on } \mathbb{S}^n \times (0, T^*),$$

where $b(t) \in C([0, T^*))$. Then there exists a positive constant $C > 0$ depending only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ and $\|v(\cdot, 0)\|_{C^3(\mathbb{S}^n)}$ such that

$$\max_{\mathbb{S}^n} v(\cdot, t) \leq C \min_{\mathbb{S}^n} v(\cdot, t),$$

for any fixed $t \in (0, T^*)$.

Proof. As mentioned in the introduction, this idea is essentially due to Ye [72]. We will show that

$$\sup_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}^n} v|}{|v|} \leq C \quad \text{for all } s \in (0, T^*).$$

Let $q_0 \in \mathbb{S}^n$. Without loss of generality, we may assume that q_0 is the north pole. Consider the inverse of the stereographic projection from the north pole $F : \mathbb{R}^n \rightarrow \mathbb{S}^n$:

$$F(x_1, \dots, x_n) = \left(\frac{2x}{1+x^2}, \frac{x^2-1}{x^2+1} \right).$$

We also denote $G : \mathbb{R}^n \rightarrow \mathbb{S}^n$ as the inverse of the stereographic projection from the south pole, namely $G(x) = F(x/|x|^2)$. Let

$$u(x, s) = \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2\sigma}{2}} v(F(x), s), \quad \bar{u}(x, s) = \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2\sigma}{2}} v(G(x), s).$$

Then $u, \bar{u} \in C^{3,1}(\mathbb{R}^n \times [0, T^*))$ and both satisfy

$$\frac{\partial u^N}{\partial t} = \Delta^\sigma u + b(t)u^N, \quad \text{on } \mathbb{R}^n \times [0, T^*). \quad (30)$$

$u(\cdot, s)$ has a Taylor expansion “at infinity” of the form

$$u(x, s) = \frac{2^{(n-2\sigma)/2}}{|x|^{n-2\sigma}} \left(a_0 + \frac{a_i x_i}{|x|^2} + \left(a_{ij} - \frac{n-2\sigma}{2} \delta_{ij} \right) \frac{x_i x_j}{|x|^4} + O(|x|^{-3}) \right).$$

Similarly partial derivatives of $u(\cdot, t)$ have a Taylor expansion “at infinity” of the form

$$\begin{aligned} \frac{\partial u}{\partial x_i}(x, s) &= 2^{\frac{n-2\sigma}{2}} \left(-\frac{n-2\sigma}{|x|^{n-2\sigma+2}} x_i \left(a_0 + \frac{a_j x_j}{|x|^2} \right) + \frac{a_i}{|x|^{n-2\sigma+2}} - \frac{2x_i a_j x_j}{|x|^{n-2\sigma+4}} \right) \\ &\quad + O(|x|^{-(n-2\sigma+3)}). \end{aligned}$$

Here

$$\begin{aligned} a_0(s) &= v(q_0, s), \\ a_i(s) &= \frac{\partial(v(\cdot, s) \circ G)}{\partial x_i}(0), \\ a_{ij}(s) &= \frac{\partial^2(v(\cdot, s) \circ G)}{2\partial x_i \partial x_j}(0). \end{aligned}$$

Let $y_i(s) = (n - 2\sigma)^{-1}a_i(s)/a_0(s)$, and $y(s) = (y_1(s), \dots, y_n(s))$. Then

$$u(x + y, s) = \frac{2^{\frac{n-2\sigma}{2}}}{|x|^{n-2\sigma}} \left(a_0 + \frac{\tilde{a}_{ij}x_ix_j}{|x|^4} + o(|x|^{-2}) \right) \quad (31)$$

and

$$\frac{\partial u}{\partial x_i}(x + y, s) = -\frac{(n - 2\sigma)a_0x_i}{|x|^{n-2\sigma+2}} + O(|x|^{-(n-2\sigma+3)}) \quad (32)$$

where $\tilde{a}_{ij} = a_{ij} - \frac{n-2\sigma}{2}\delta_{ij} - \frac{a_ia_j}{a_0}$. We only need to show that there exist a positive constant C depending only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ and $\|v(\cdot, 0)\|_{C^3(\mathbb{S}^n)}$ such that

$$|y(s)| \leq C \quad \text{for all } 0 \leq s < T^*.$$

Fix $T \in (0, T^*)$. After a rotation and a reflection, we may assume that $y_n(T) = \max_i |y_i(T)|$. From the Taylor expansion of u and ∇u for $s = 0$, we see that (e.g., Lemma 4.2 in [35]) there exists a $\lambda_0 > 0$, which depends only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ and $\|v(\cdot, 0)\|_{C^3(\mathbb{S}^n)}$, such that for any $\lambda > \lambda_0$,

$$u(x, 0) > u(x^\lambda, 0) \quad \text{for } x_n < \lambda$$

where $x^\lambda = (x_1, \dots, x_{n-1}, 2\lambda - x_n)$. Denote $u^\lambda(x, s) = u(x^\lambda, s)$. By Proposition 2.3, we have

$$u(x, s) > u^\lambda(x, s) \quad \text{for all } s \in [0, T], \quad x_n < \lambda, \quad \lambda \geq \lambda_0. \quad (33)$$

We claim that

$$\max_{0 \leq s \leq T} y_n(s) < \lambda_0.$$

If not, there exists $\bar{s} \in (0, T]$ such that $y_n(\bar{s}) = \max_{0 \leq s \leq T} y_n(s) \geq \lambda_0$. Thus we can set $\lambda = y_n(\bar{s})$ in (33), namely,

$$u(x, s) > u^\lambda(x, s) \quad \text{for all } s \in [0, T], \quad x_n < \lambda = y_n(\bar{s}).$$

Let $\tilde{u}(x, s) = u(x + y_n(\bar{s}), s)$, then

$$\tilde{u}(x', x_n, s) > \tilde{u}(x', -x_n, s) \quad \text{for all } s \in [0, T], \quad x_n < 0.$$

Let $\tilde{u}_1(x, s) = \frac{1}{|x|^{n-2\sigma}} \tilde{u}(\frac{x}{|x|^2}, s)$. Then $\tilde{u}_1(x', x_n, s)$ and $\tilde{u}_1(x', -x_n, s)$ satisfy (30) and

$$\tilde{u}_1(x', x_n, s) > \tilde{u}_1(x', -x_n, s) \quad \text{for all } s \in [0, T], \quad x_n < 0.$$

By Proposition 2.3

$$\left| \frac{\partial(\tilde{u}_1(x', x_n, s) - \tilde{u}_1(x', -x_n, s))}{\partial x_n} \right|_{(x,s)=(0,\bar{s})} < 0,$$

i.e., $(\partial \tilde{u}_1 / \partial x_n)(0, \bar{s}) < 0$. This contradicts (31). Hence $\max_{0 \leq s \leq T} y_n(s) < \lambda_0$, which implies $y_n(T) < \lambda_0$. Since λ_0 is independent of s , we have $|y(s)| \leq \lambda_0$ for all $0 \leq s < T^*$. Moreover λ_0 is independent of the choice of q_0 , we conclude that

$$\sup_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}^n} v|}{|v|} \leq C \quad \text{for all } s \in (0, T^*).$$

For each t , integrating the above inequality along a shortest geodesic between a maximum point and a minimum point of $v(\cdot, t)$ yields

$$\max_{\mathbb{S}^n} v(\cdot, t) \leq C \min_{\mathbb{S}^n} v(\cdot, t).$$

where C depends only on $n, \sigma, \inf_{\mathbb{S}^n} v(\cdot, 0)$ and $\|v(\cdot, 0)\|_{C^3(\mathbb{S}^n)}$. \square

4 Existence and convergence of a fractional Yamabe flow

4.1 Schauder estimates

For an open set $\Omega \subset \mathbb{R}^n$ and $\gamma \in (0, 1)$, $C^\gamma(\Omega)$ denotes the standard Hölder space over Ω , with the norm

$$|v|_{\gamma; \Omega} := |v|_{0; \Omega} + [v]_{\gamma; \Omega} := \sup_{\Omega} |v(\cdot)| + \sup_{x_1 \neq x_2, x_1, x_2 \in \Omega} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\gamma},$$

For simplicity, we use $C^\gamma(\Omega)$ to denote $C^{[\gamma], \gamma - [\gamma]}(\Omega)$ when $1 < \gamma \notin \mathbb{N}$ (the set of positive integers), where $[\gamma]$ is the integer part of γ . Since the operator $\partial_t + (-\Delta)^\sigma$ is invariant under the scaling $(x, t) \rightarrow (cx, c^{2\sigma}t)$ with $c > 0$, we introduce the fractional parabolic distance as

$$\rho(X_1, X_2) = (|x_1 - x_2|^2 + |t_1 - t_2|^{1/\sigma})^{1/2},$$

where $X_1 = (x_1, t_2), X_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$. For a measurable function u defined in a Borel set $Q \subset \mathbb{R}^{n+1}$ and $0 < \alpha < \min(1, 2\sigma)$, we define

$$[u]_{\alpha, \frac{\alpha}{2\sigma}; Q} = \sup_{X_1 \neq X_2, X_1, X_2 \in Q} \frac{|u(X_1) - u(X_2)|}{\rho(X_1, X_2)^\alpha},$$

and

$$|u|_{\alpha, \frac{\alpha}{2\sigma}; Q} = |u|_{0; Q} + [u]_{\alpha, \frac{\alpha}{2\sigma}; Q},$$

where $|u|_{0; Q} = \sup_{X \in Q} |u(X)|$. We denote $C^{\alpha, \frac{\alpha}{2\sigma}}(Q)$ as the space of all measurable functions u for which $|u|_{\alpha, \frac{\alpha}{2\sigma}; Q} < \infty$. Let $Q_T = \mathbb{R}^n \times (0, T]$, $T \in (0, \infty)$. For $2\sigma + \alpha \notin \mathbb{N}$ and $0 < \alpha < \min(1, 2\sigma)$, we say $u \in C^{2\sigma + \alpha, 1 + \frac{\alpha}{2\sigma}}(Q_T)$ if

$$[u]_{2\sigma + \alpha, 1 + \frac{\alpha}{2\sigma}; Q_T} := [u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} + [(-\Delta)^\sigma u]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} < \infty$$

and

$$|u|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} := |u|_{0; Q_T} + |u_t|_{0; Q_T} + |(-\Delta)^\sigma u|_{0; Q_T} + [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} < \infty.$$

Then $\mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_T)$ is a Banach space equipped with the norm $|\cdot|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T}$.

Consider the following Cauchy problem

$$\begin{cases} a(x, t)u_t + (-\Delta)^\sigma u + b(x, t)u = f(x, t), & \text{in } Q_T, \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases} \quad (34)$$

where $\lambda^{-1} \leq a(x, t) \leq \lambda$ for some constant $\lambda \geq 1$.

Lemma 4.1. *Suppose $b(x, t)$ is bounded in Q_1 . Let $u \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_1)$ satisfy*

$$\begin{cases} a(x, t)u_t + (-\Delta)^\sigma u + b(x, t)u \leq 0, & \text{in } Q_1, \\ u(x, 0) \leq 0, & \text{in } \mathbb{R}^n, \end{cases}$$

then $u \leq 0$ in Q_1 .

Proof. Without loss of generality we may assume that $b(x, t) \geq 1$ as before. Let $\eta(x)$ be a smooth cut-off function supported in $B_2 \in \mathbb{R}^n$ and equal to 1 in B_1 . Let $\eta_R(\cdot) = \eta(\cdot/R)$ and $v = \eta_R u$. Then

$$av_t + (-\Delta)^\sigma v + b(x, t)v \leq \langle u, \eta_R \rangle + u(-\Delta)^\sigma \eta_R,$$

where

$$\langle u, \eta \rangle = c(n, \sigma) \int_{\mathbb{R}^n} \frac{(u(x, t) - u(y, t))(\eta(x, t) - \eta(y, t))}{|x - y|^{n+2\sigma}} dy. \quad (35)$$

If u is positive somewhere in Q_1 , then we can choose R as large as we want such that v attains its positive maximum value in Q_1 at $(x_0, t_0) \in B_R \times (0, 1]$. It is clear that $a(x_0, t_0)v_t(x_0, t_0) + (-\Delta)^\sigma v(x_0, t_0) \geq 0$. Since $b \geq 1$, we have

$$\sup_{B_R \times (0, 1]} u \leq v(x_0, t_0) \leq \sup_{Q_1} |\langle u, \eta_R \rangle + u(-\Delta)^\sigma \eta_R| \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

This finishes the proof of this Lemma. \square

Proposition 4.1. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Suppose that $a(x, t)$, $b(x, t)$, $f(x, t) \in C^{\alpha, \frac{\alpha}{2\sigma}}(Q_1)$ and $u_0(x) \in C^{2\sigma+\alpha}(\mathbb{R}^n)$, then there exists a unique solution $u \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_1)$ of (34). Moreover there exists a constant $C > 0$ depending only on $n, \sigma, \lambda, \alpha$, $|a|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}$ and $|b|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}$ such that*

$$|u|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} \leq C(|u_0|_{2\sigma+\alpha; \mathbb{R}^n} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}). \quad (36)$$

Proof. By Lemma 4.1, there exists $C > 0$ depending only on $\lambda, |b|_{L^\infty(Q_1)}$ such that

$$|u|_{0;Q_1} \leq C(|u_0|_{0;\mathbb{R}^n} + |f|_{0;Q_1}). \quad (37)$$

Uniqueness follows immediately. In the following, we will show a priori estimates (36). By (37) and some interpolation inequalities in Lemma B.1 we only need to show, instead of (36),

$$[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_1} \leq C(|u_0|_{2\sigma+\alpha;\mathbb{R}^n} + |f|_{\alpha,\frac{\alpha}{2\sigma};Q_1}). \quad (38)$$

First of all, (38) holds provided $a = 1, b = 0$ (see, e.g., [56]). It could be easily extended to the case that a is a positive constant. For the general case, we use the “freezing coefficients” method (see, e.g., [53]).

Fix a small $\delta > 0$, which will be specified later. We can find two points $X_1, X_2 \in Q_1$ such that

$$\frac{|u_t(X_1) - u_t(X_2)|}{\rho(X_1, X_2)^\alpha} \geq \frac{1}{2}[u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1}.$$

If $\rho(X_1, X_2) > \delta$, then

$$[u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1} \leq 4\delta^{-\alpha}|u_t|_{0;Q_1}.$$

It follows from Lemma B.1 that, for any small $\varepsilon_0 > 0$,

$$[u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1} \leq \varepsilon_0[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_1} + C_0|u|_{0;Q_1}, \quad (39)$$

where $C_0 > 0$ depending on $n, \sigma, \alpha, \varepsilon_0, \delta$.

If $\rho(X_1, X_2) \leq \delta$, take a cut-off function $\eta(X) \in C^\infty(\mathbb{R}^{n+1})$ such that $\eta(X) = 1$ for $\rho(X, X_1) \leq \delta$, $\eta(X) = 0$ for $\rho(X, X_1) \geq 2\delta$. By the estimates of solutions of (34) with a being a positive constant and $b \equiv 0$, we have

$$\begin{aligned} [u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1} &\leq 2 \frac{|u_t(X_1) - u_t(X_2)|}{\rho(X_1, X_2)^\alpha} \leq 2[u\eta]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_1} \\ &\leq C_1(|a(X_1)(u\eta)_t + (-\Delta)^\sigma(u\eta)|_{\alpha,\frac{\alpha}{2\sigma};Q_1} + |u_0\eta|_{2\sigma+\alpha;\mathbb{R}^n} + |u\eta|_{0;Q_1}), \end{aligned}$$

where $C_1 > 0$ is independent of δ . Note that

$$\begin{aligned} &a(X_1)(u\eta)_t + (-\Delta)^\sigma(u\eta) \\ &= \eta(a(X)u_t + (-\Delta)^\sigma u) + \eta(a(X_1) - a(X))u_t + a(X_1)u\eta_t - \langle u, \eta \rangle + u(-\Delta)^\sigma \eta \\ &= \eta(f - bu) + \eta(a(X_1) - a(X))u_t + a(X_1)u\eta_t - \langle u, \eta \rangle + u(-\Delta)^\sigma \eta, \end{aligned}$$

where $\langle u, \eta \rangle$ is defined in (35). Since $|\eta(X)(a(X_1) - a(X))| \leq [a]_{\alpha,\frac{\alpha}{2\sigma};Q_1} \delta^\alpha$, making use of Lemma B.1 again, we have

$$\begin{aligned} [u_t]_{\alpha,\frac{\alpha}{2\sigma};Q_1} &\leq C_1 \delta^\alpha [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_1} + C(\delta)(|u|_{0;Q_1} + |f|_{\alpha,\frac{\alpha}{2\sigma};Q_1}) \\ &\quad + C_1 |\langle u, \eta \rangle|_{\alpha,\frac{\alpha}{2\sigma};Q_1} + C_1 |u_0\eta|_{2\sigma+\alpha;\mathbb{R}^n}. \end{aligned} \quad (40)$$

Hence from (62) in Lemma B.3, (40) and (39), we can conclude that

$$[u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq (C_1 \delta^\alpha + \varepsilon_0) [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} + C(\delta) (|u|_{0, Q_1} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1} + |u_0|_{2\sigma+\alpha; \mathbb{R}^n}). \quad (41)$$

Since

$$u_t + (-\Delta)^\sigma u = (1 - a)u_t - bu + f,$$

we see that

$$[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} \leq C([u_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} + |u|_{0, Q_1} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1} + |u_0|_{2\sigma+\alpha; \mathbb{R}^n}), \quad (42)$$

where $C > 0$ depending only on $n, \sigma, \lambda, \alpha, \|a\|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}$ and $\|a, b\|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}$. Then (36) follows from (37), (42) and (41) by choosing sufficiently small δ and ε_0 .

Finally, the existence of solutions of (34) follows from standard continuity method. \square

Remark 4.1. *Cauchy problems for non-local operators and pseudo-differential operators in different spaces have been studied, e.g., in [52], [58], [59], [60] and references therein.*

Remark 4.2. *Observe that in the proof of the above proposition the only place we use the uniform lower and upper bounds of $a(x)$ is that at X_1 , that is $\frac{1}{\lambda} \leq a(X_1) \leq \lambda$. This observation will be used in the proof of Proposition 4.2.*

Remark 4.3. *One can also obtain the estimates in Q_T by considering the scaled function $\tilde{u}(x, t) := u(T^{1/2\sigma}x, Tt)$.*

For $\gamma \in (0, 1)$, $C^\gamma(\mathbb{S}^n)$ denotes the standard Hölder space over \mathbb{S}^n , with norm

$$|v|_{\gamma; \mathbb{S}^n} := |v|_{0; \mathbb{S}^n} + [v]_{\gamma; \mathbb{S}^n} := \sup_{\mathbb{S}^n} |v(\cdot)| + \sup_{\xi_1 \neq \xi_2, \xi_1, \xi_2 \in \mathbb{S}^n} \frac{|v(\xi_1) - v(\xi_2)|}{|\xi_1 - \xi_2|^\gamma},$$

where $|\xi_1 - \xi_2|$ is understood as the Euclidean distance from ξ_1 to ξ_2 in \mathbb{R}^{n+1} . For simplicity, we use $C^\gamma(\mathbb{S}^n)$ to denote $C^{[\gamma], \gamma - [\gamma]}(\mathbb{S}^n)$ when $1 < \gamma \notin \mathbb{N}$, where $[\gamma]$ is the integer part of γ . For $Y_1 = (\xi_1, t_1), Y_2 = (\xi_2, t_2) \in \mathbb{S}^n \times (0, \infty)$ we denote

$$\rho(Y_1, Y_2) = (|\xi_1 - \xi_2|^2 + |t_1 - t_2|^{1/\sigma})^{1/2}.$$

We still assume that $0 < \alpha < \min(1, 2\sigma)$. Let $Q_T = \mathbb{S}^n \times (0, T]$ for $T > 0$. We say $v \in C^{\alpha, \frac{\alpha}{2\sigma}}(Q_T)$ if

$$|v|_{\alpha, \frac{\alpha}{2\sigma}; Q_T} = |v|_{0; Q_T} + [v]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} := \sup_{Y \in Q_T} v(Y) + \sup_{Y_1 \neq Y_2, Y_1, Y_2 \in Q_T} \frac{|v(Y_1) - v(Y_2)|}{\rho(Y_1, Y_2)^\alpha} < \infty,$$

and $v \in C^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_T)$ if

$$[v]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} := [v_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} + [P_\sigma(v)]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} < \infty$$

and

$$|v|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; \mathcal{Q}_T} := |v|_{0; \mathcal{Q}_T} + |v_t|_{0; \mathcal{Q}_T} + |P_\sigma(v)|_{0; \mathcal{Q}_T} + [v]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; \mathcal{Q}_T} < \infty.$$

Then $\mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_T)$ is a Banach space equipped with the norm $|\cdot|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; \mathcal{Q}_T}$.

Proposition 4.2. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Let $a(\xi, t)$, $b(\xi, t)$, $f(\xi, t) \in C^{\alpha, \frac{\alpha}{2\sigma}}(\mathcal{Q}_1)$, $v_0 \in C^{2\sigma+\alpha}(\mathbb{S}^n)$ and $\lambda^{-1} \leq a(\xi, t) \leq \lambda$ for some $\lambda \geq 1$. Then there exists a unique function $v \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathcal{Q}_1)$ such that*

$$\begin{cases} av_t + P_\sigma(v) + bv = f, & \text{in } \mathcal{Q}_1, \\ v(y, 0) = v_0(y). \end{cases} \quad (43)$$

Moreover there exists a constant C depending only on $n, \sigma, \lambda, \alpha, |a|_{\alpha, \frac{\alpha}{2\sigma}; \mathcal{Q}_1}$ and $|b|_{\alpha, \frac{\alpha}{2\sigma}; \mathcal{Q}_1}$ such that

$$|v|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; \mathcal{Q}_1} \leq C(|v_0|_{2\sigma+\alpha; \mathbb{S}^n} + |f|_{\alpha, \frac{\alpha}{2\sigma}; \mathcal{Q}_1}). \quad (44)$$

Proof. Uniqueness follows from the maximum principle. We only need to show a priori estimate (44), from which the existence of solution of (43) follows by the standard continuity method.

Choose $Y_1 = (\xi_1, t_1), Y_2 = (\xi_2, t_2) \in \mathbb{S}^n \times (0, T)$ such that

$$\frac{|v_t(Y_1) - v_t(Y_2)|}{\rho(Y_1, Y_2)^\alpha} \geq \frac{1}{2} [v_t]_{\alpha, \frac{\alpha}{2\sigma}; \mathcal{Q}_1}. \quad (45)$$

Without loss of generality we may assume that ξ_1, ξ_2 are on the south hemisphere. Let $F(x)$ be the inverse of stereographic projection from the north pole and

$$u(x, t) = \left(\frac{2}{1 + |x|^2} \right)^{\frac{n-2\sigma}{2}} v(F(x), t).$$

There exist $x_1, x_2 \in B(0, 1)$ such that $Y_1 = (F(x_1), t_1), Y_2 = (F(x_2), t_2)$. We denote $X_1 = (x_1, t_1), X_2 = (x_2, t_2)$. By (45) there exists a constant C depending only n, σ, α such that

$$[u_t]_{\alpha, \frac{\alpha}{2\sigma}; \mathcal{Q}_1} \leq C|v_t|_{0; \mathcal{Q}_1} + C|u_t|_{0; \mathcal{Q}_1} + C \frac{|u_t(X_1) - u_t(X_2)|}{\rho(X_1, X_2)^\alpha}.$$

Note that u satisfies (34) with a, b, f replaced by

$$\left(\frac{2}{1 + |x|^2} \right)^{2\sigma} a(F(x), t), \left(\frac{2}{1 + |x|^2} \right)^{2\sigma} b(F(x), t), \left(\frac{2}{1 + |x|^2} \right)^{\frac{n+2\sigma}{2}} f(F(x), t).$$

In view of Remark 4.2 and the arguments in the proof of Proposition 4.1, we conclude that

$$[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; \mathcal{Q}_1} \leq C(|v_0|_{2\sigma+\alpha; \mathbb{S}^n} + |v|_{0; \mathcal{Q}_1} + |v_t|_{0; \mathcal{Q}_1} + |f|_{\alpha, \frac{\alpha}{2\sigma}; \mathcal{Q}_1}).$$

Hence, together with (45) and interpolation inequalities in Lemma B.2, we have

$$[v_t]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq C(|v_0|_{2\sigma+\alpha; \mathbb{S}^n} + |v|_{0; Q_1} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1}). \quad (46)$$

It follows from the maximum principle that $|v|_{0; Q_1} \leq C(|v_0|_{2\sigma+\alpha; \mathbb{S}^n} + |f|_{\alpha, \frac{\alpha}{2\sigma}; Q_1})$. Hence (44) follows from (46), (43) and some inequalities in Lemma B.2. \square

Corollary 4.1. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Let $a(\xi, t)$, $b(\xi, t)$, $f(\xi, t) \in C^{\alpha, \frac{\alpha}{2\sigma}}(Q_3)$, $\lambda^{-1} \leq a(\xi, t) \leq \lambda$ for some $\lambda \geq 1$. Suppose that $v \in C^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_3)$ satisfies*

$$av_t + P_\sigma(v) + bv = f, \quad \text{in } Q_3.$$

Then there exists a constant C depending only on $n, \sigma, \lambda, \alpha, |a|_{\alpha, \frac{\alpha}{2\sigma}; \mathbb{S}^n \times [1, 3]}$ and $|b|_{\alpha, \frac{\alpha}{2\sigma}; \mathbb{S}^n \times [1, 3]}$ such that

$$|v|_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; \mathbb{S}^n \times [2, 3]} \leq C(|v|_{\alpha, \frac{\alpha}{2\sigma}; \mathbb{S}^n \times [1, 3]} + |f|_{\alpha, \frac{\alpha}{2\sigma}; \mathbb{S}^n \times [1, 3]}).$$

Proof. Let $\eta(t)$ be a smooth cut-off function defined on \mathbb{R} such that $\eta(t) = 0$ when $t \leq 4/3$ and $\eta(t) = 1$ when $t \geq 5/3$. Then $\tilde{v} := \eta v$ satisfies

$$\begin{cases} a\tilde{v}_t + P_\sigma(\tilde{v}) + b\tilde{v} = f\eta + av\eta_t, & \text{in } \mathbb{S}^n \times [1, 3], \\ \tilde{v}(\cdot, 1) = 0. \end{cases}$$

The Corollary follows immediately from Proposition 4.2. \square

4.2 Short time existence

Proposition 4.3. *Let $0 < \alpha < \min(1, 2\sigma)$ such that $2\sigma + \alpha$ is not an integer. Let $v_0 \in C^{2\sigma+\alpha}(\mathbb{S}^n)$ and $v_0 > 0$ in \mathbb{S}^n . Then there exist a small positive constant T_* depending only on $n, \sigma, \alpha, \inf_{\mathbb{S}^n} v_0, |v_0|_{2\sigma+\alpha; \mathbb{S}^n}$ and a unique positive solution $v \in C^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times [0, T_*])$ of (8) in $\mathbb{S}^n \times (0, T_*]$ with $v(\cdot, 0) = v_0$. Furthermore, v is smooth in $\mathbb{S}^n \times (0, T_*)$.*

Proof. By a scaling argument in the time variable, we only need to show the short time existence of

$$\begin{cases} \frac{\partial v^N}{\partial t} = -P_\sigma(v), \\ v(\cdot, 0) = v_0. \end{cases}$$

We shall use the Implicit Function Theorem. By Proposition 4.2, there exists a function $w \in C^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, 1])$ such that

$$\begin{cases} Nv_0^{N-1}w_t = -P_\sigma(w), & \text{in } \mathbb{S}^n \times (0, 1], \\ w(\cdot, 0) = v_0, \end{cases}$$

and for any small positive constant ε_0 , we have $\|w(\cdot, t) - v_0\|_{C^{2\sigma+\alpha}(\mathbb{S}^n)} \leq \varepsilon_0$ provided $t \leq T_{\varepsilon_0}$. Here T_{ε_0} is a positive constant depending on ε_0 . Hence we assume that $w > 0$ in \mathbb{S}^n .

Denote

$$\mathcal{X} = \{\varphi \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, T_{\varepsilon_0}]) : \varphi(\cdot, 0) = 0\},$$

and

$$\mathcal{Y} = C^{\alpha, \frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, T_{\varepsilon_0}]).$$

Define $\mathcal{F}(v) := N|v|^{N-1}\frac{\partial v}{\partial t} + P_\sigma(v)$ for $v \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(\mathbb{S}^n \times (0, T_{\varepsilon_0}])$, and

$$L : \mathcal{X} \rightarrow \mathcal{Y}, \quad \varphi \mapsto \mathcal{F}(w + \varphi) - \mathcal{F}(w).$$

Note that $L(0) = 0$,

$$L'(0)\varphi = Nw^{N-1}\varphi_t + P_\sigma(\varphi) + N(N-1)w^{N-2}w_t\varphi, \quad \forall \varphi \in \mathcal{X}.$$

Choosing ε_0 sufficiently small, it follows from Proposition 4.2 that $L'(0) : \mathcal{X} \rightarrow \mathcal{Y}$ is invertible.

By the Implicit Function Theorem, there exists a positive constant $\delta > 0$ such that for any $\phi \in \mathcal{Y}$ with $\|\phi\|_{\mathcal{Y}} \leq \delta$ there exists a unique solution $\varphi \in \mathcal{X}$ of the equation

$$L(\varphi) = \phi.$$

Let $T_* > 0$ be small. Pick a cut off function $0 \leq \eta(t) \leq 1$ in \mathbb{R}_+ satisfying $\eta(t) = 1$ for $s \leq T_*$ and $\eta(t) = 0$ if $s \geq 2T_*$. It is easy to see that

$$\|\eta(t)\mathcal{F}(w)\|_{\mathcal{Y}} \leq \delta,$$

provided T_* is sufficiently small. Therefore, there exists a function $\varphi \in \mathcal{X}$ such that

$$L(\varphi) = -\eta(t)\mathcal{F}(w).$$

Thus $v := w + \varphi$ satisfies $v(\cdot, 0) = v_0$ and

$$\mathcal{F}(w + \varphi) = 0, \quad \text{in } \mathbb{S}^n \times (0, T_*].$$

Moreover, v is positive if T_* is small enough. The smoothness of v follows from Corollary 4.1 and bootstrap arguments. \square

4.3 Long time existence and convergence

Proposition 4.4. *Let v be a positive smooth solution of (8) in $\mathbb{S}^n \times (0, 3]$ and satisfy $\Lambda^{-1} \leq v(y, t) \leq \Lambda$ for all $(y, t) \in \mathbb{S}^n \times (0, 3]$ with some positive constant Λ . Then for any positive integer k ,*

$$\|v\|_{C^k(\mathbb{S}^n \times [2, 3])} \leq C \tag{47}$$

where $C > 0$ depends only on n, σ, k, Λ , and $r_\sigma^{g(1)}$.

Proof. We first observe that $r_\sigma^{g(t)}$ is decreasing in t , and is lower bounded away from 0 by Sobolev inequalities (see, e.g., [5]). Hence through a scaling argument in t , we may assume that v satisfies the equation $\frac{\partial v^N}{\partial t} = -P_\sigma(v)$ instead of (8). By the Hölder estimates in [2] (see also Theorem 9.2 in [28]), there exists some $\beta \in (0, \min(1, 2\sigma))$ such that

$$|v|_{\beta, \frac{\beta}{2\sigma}; \mathbb{S}^n \times [1, 3]} \leq C(n, \sigma, \beta, \Lambda).$$

The Proposition follows from Corollary 4.1 and bootstrap arguments. \square

Proof of Theorem 1.1. By Proposition 4.3, we have a unique positive smooth solution of (7) on a maximum time interval $[0, T^*)$. Since the flow preserves the volume of the sphere, the Harnack inequality in Theorem 3.1 implies that $v(x, t)$ is uniformly bounded from above and away from zero. Proposition 4.4 yields smooth estimates for v on $\mathbb{S}^n \times [\min(1, T^*/2), T^*)$. It follows that $T^* = \infty$, since otherwise by Proposition 4.3 we can extend v beyond T^* . Moreover there exists $v_\infty \in C^\infty(\mathbb{S}^n)$ and a sequence $v(t_j)$ such that $v(t_j)$ converges smoothly to v_∞ . By Theorem A.4 in the Appendix, $v(t)$ converges smoothly to v_∞ , i.e. there exists a smooth metric g_∞ on \mathbb{S}^n such that $g(t)$ converges smoothly to g_∞ . The formula for the gradient of the total σ curvature gives

$$\frac{dS}{dt} = -\frac{n-2\sigma}{2n} (\text{vol}_g(\mathbb{S}^n))^{\frac{2\sigma-n}{n}} \int_{\mathbb{S}^n} (R_\sigma^g - r_\sigma^g)^2 d\text{vol}_g.$$

Thus

$$\int_0^\infty \int_{\mathbb{S}^n} (R_\sigma^g - r_\sigma^g)^2 d\text{vol}_g < \infty,$$

which implies that $R_\sigma^{g_\infty}$ is a positive constant. \square

5 Extinction profile of a fractional porous medium equation

Let $u(x, t)$ be the solution of (11) and $T > 0$ be its extinction time. Since u_0 is not identically zero, it is proved in [28] that $u(x, t) > 0$ in $\mathbb{R}^n \times (0, T)$ and $u(x, t) \in C^\alpha(\mathbb{R}^n \times (0, T))$ for some $\alpha \in (0, 1)$. We define $v(F(x), s)$ for all $x \in \mathbb{R}^n$ and all $s \geq 0$ as

$$v(F(x), s) := \left(\frac{1 + |x|^2}{2} \right)^{\frac{n-2\sigma}{2}} (T - t)^{-m/(1-m)} u(x, t)^m \Big|_{t=T(1-e^{-s})}, \quad (48)$$

where $F : \mathbb{R}^n \rightarrow \mathbb{S}^n$ is the inverse of stereographic projection from the north pole and $m = \frac{n-2\sigma}{n+2\sigma}$. By the assumption of u_0 , we have $v(\cdot, 0) \in C^2(\mathbb{S}^n)$. It follows from Proposition 4.3 that, there exist an $s^* > 0$ and a unique positive function $\tilde{v} \in C^\infty(\mathbb{S}^n \times (0, s^*))$ satisfies

$$\frac{\partial \tilde{v}^N}{\partial s} = -P_\sigma(\tilde{v}) + \frac{1}{1-m} \tilde{v}^N \quad (49)$$

and $\tilde{v}_0 = v(\cdot, 0)$. On the other hand, $\tilde{u}(x, t)$, which is defined by \tilde{v} through (48), satisfies (11). By the uniqueness theorem on the solution of (11) in [28], $v \equiv \tilde{v}$ in $\mathbb{S}^n \setminus \{\mathcal{N}\} \times (0, s^*)$, and hence v can be extended to a positive and smooth function in $\mathbb{S}^n \times (0, s^*)$.

Our first goal is that v defined by relation (48) is positive and smooth in $\mathbb{S}^n \times (0, \infty)$. Secondly, we will show that v converges to a steady solution of (49). In summary, we will show the following theorem in terms of v .

Theorem 5.1. *Let v be defined by relation (48). Then v is positive and smooth in $\mathbb{S}^n \times (0, \infty)$. Moreover there is a unique positive solution \bar{v} of*

$$-P_\sigma(\bar{v}) + \frac{1}{1-m}\bar{v}^N = 0 \quad (50)$$

such that

$$\|v(y, s) - \bar{v}(y)\|_{C^3(\mathbb{S}^n)} \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

The proof of Theorem 5.1 here uses similar strategies to that of [29]. To prove convergence of $v(\cdot, t)$ we need the following universal estimates

Proposition 5.1. *Let v be defined by relation (48). There exist positive constants β_1, β_2 such that*

$$\beta_1 \leq v(y, s) \leq \beta_2$$

for all $y \in \mathbb{S}^n$, $s^*/2 \leq s < +\infty$. Hence $v \in C^\infty(\mathbb{S}^n \times (s^*/2, \infty))$.

Proof. Step1: We show that if s_0 is such that v is positive and smooth in $\mathbb{S}^n \times (s^*/2, s_0)$, then there is a positive constant κ_1 , independent of s_0 , such that for all $s \in (s^*/2, s_0)$

$$\max_{\mathbb{S}^n} v(\cdot, s) > \kappa_1. \quad (51)$$

Let us argue by contradiction. If this is not true, then for every small $\varepsilon > 0$, there is an s_ε such that $s_0 > s_\varepsilon > s^*/2$ and $v(y, s_\varepsilon) < \varepsilon$ for all $y \in \mathbb{S}^n$. Given $\varepsilon > 0$, consider

$$U(x, t) = K^{1/m} [(1 + s_\varepsilon - \log T + \log(T - t))(T - t)]_+^{\frac{1}{1-m}} \left(\frac{2}{1 + |x|^2} \right)^{\frac{n+2\sigma}{2}},$$

where K will be chosen later. Direct computations yield that

$$\begin{aligned} & U_t - (\Delta)^\sigma U^{\frac{n-2\sigma}{n+2\sigma}} \\ &= K^{\frac{1}{m}} [(1 + s_\varepsilon - \log T + \log(T - t))(T - t)]_+^{\frac{m}{1-m}} \left(\frac{2}{1 + |x|^2} \right)^{\frac{n+2\sigma}{2}} \\ & \quad \cdot \left(\log T - \log(T - t) - 2 - s_\varepsilon + P_\sigma(1)K^{1-1/m} \right), \end{aligned}$$

where we used that $(-\Delta)^\sigma \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2\sigma}{2}} = P_\sigma(1) \left(\frac{2}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}}$ with $P_\sigma(1)$ given in (4).

Let t_ε be that $s_\varepsilon - \log T + \log(T - t_\varepsilon) = 0$. We choose K small such that $P_\sigma(1)K^{\frac{m-1}{m}} > 2$ and let $\varepsilon = K$. Since $v(y, s_\varepsilon) < \varepsilon$,

$$u(x, t_\varepsilon) < \varepsilon^{1/m} (T - t_\varepsilon)^{\frac{1}{1-m}} \left(\frac{2}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}} = U(x, t_\varepsilon).$$

For $t > t_\varepsilon$, $U(x, t)$ is a supersolution of (11). It follows from the comparison principle (see the proof of Theorem 6.2 in [28]) that $u(x, t) \leq U(x, t)$. But U vanishes before T . Hence u vanishes before T , which contradicts the definition of the extinction time T .

Step 2: v is strictly positive and smooth for $s^*/2 < s < \infty$.

To show this, we define

$$s_0 = \sup\{s > 0 : v \in C^{3,1}(\mathbb{S}^n \times (s^*/2, s))\}.$$

Note that $s_0 \geq s^*$. We assume that $s_0 < \infty$. Since $v \in C^{3,1}(\mathbb{S}^n \times (s^*/2, s_0))$ and v is positive, by Theorem 3.1 and step 1 we have that v is uniformly lower bounded away from 0. We define

$$U(x, t) = (M - t)_+^{1/(1-m)} k(n, \sigma) \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}},$$

where $k(n, \sigma)$ is defined in Theorem 1.2. $U(x, t)$ satisfies (11) and will be used as a barrier function. By our assumptions on u_0 , we choose sufficiently large $M > T$ such that

$$u_0(x) \leq M^{1/(1-m)} k(n, \sigma) \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}}.$$

It follows by comparison principle (Theorem 6.2 in [28]) that for all $0 < t < T$,

$$u(x, t) \leq (M - t)^{1/(1-m)} k(n, \sigma) \left(\frac{1}{1+|x|^2} \right)^{\frac{n+2\sigma}{2}}.$$

Hence for all $s^*/2 \leq s \leq s_0$

$$v(y, s) \leq \left(\frac{T + (M - T)e^s}{T} \right)^{\frac{m}{1-m}} k(n, \sigma)^m \leq \left(\frac{T + (M - T)e^{s_0}}{T} \right)^{\frac{m}{1-m}} k(n, \sigma)^m.$$

Thus v is uniformly bounded from above. Since v satisfies (49), Proposition 4.4 implies that v has a uniform limit as $s \rightarrow s_0$ which is also positive and smooth. By Proposition 4.3 v can be extended in a smooth and positive way beyond s_0 , which violates the definition of s_0 . Hence $s_0 = +\infty$.

Step 3: There is a constant $\kappa_2 = (1 + P_\sigma(1)(1 - m))^{m/(1-m)} > 0$ such that for all $s > 0$

$$\min_{\mathbb{S}^n} v(y, s) \leq \kappa_2.$$

We argue by contradiction. Suppose that there is a time $\bar{s} < \infty$ for which

$$\min_{\mathbb{S}^n} v(y, \bar{s}) > \kappa_2.$$

This implies

$$u(x, \bar{t}) \geq (T - \bar{t} + P_\sigma(1)(1 - m)(T - \bar{t}))^{1/(1-m)} \left(\frac{1}{1 + |x|^2} \right)^{\frac{n+2\sigma}{2}},$$

where $\bar{t} = T(1 - e^{-\bar{s}}) < T$. We consider a barrier function

$$U(x, t) = (T - \bar{t} + P_\sigma(1)(1 - m)(T - t))^{\frac{1}{1-m}} \left(\frac{1}{1 + |x|^2} \right)^{\frac{n+2\sigma}{2}},$$

which satisfies (11). Since $u(x, \bar{t}) \geq U(x, \bar{t})$, by the comparison principle

$$u(x, \bar{t}) \geq (T - \bar{t} + P_\sigma(1)(1 - m)(T - t))^{\frac{1}{1-m}} \left(\frac{1}{1 + |x|^2} \right)^{\frac{n+2\sigma}{2}}.$$

This contradicts the extinction time T of u .

From Steps 1, 2 and 3 we can conclude Proposition 5.1 by taking $\beta_2 = C\kappa_2$ and $\beta_1 = \kappa_1/C$ where C is the constant in Theorem 3.1 for $s_0 = \infty$. \square

Now we are in the position to prove Theorem 5.1. Let J be the functional defined as

$$J(z) = \frac{1}{2} \int_{\mathbb{S}^n} z P_\sigma(z) - \frac{1}{(1 - m)(N + 1)} \int_{\mathbb{S}^n} z^{N+1}.$$

Direct computations yield

Lemma 5.1. *Let $v(x, s)$ satisfy (49). Then*

$$\frac{d}{ds} J(v(\cdot, s)) = -N \int_{\mathbb{S}^n} v^{N-1} (v_s)^2 \leq 0.$$

The above Lemma indicates that the functional is decreasing in time. The next Lemma states that this functional is always nonnegative, and hence $\lim_{s \rightarrow \infty} J(v(\cdot, s))$ exists.

Lemma 5.2. *$J(v(\cdot, s)) \geq 0$ for all $s > 0$.*

Proof. The proof is similar to that of Lemma 6.1 in [29], which is included here for completeness. We argue by contradiction. Assume that for certain $0 < s_0 < \infty$ one has $J(v(\cdot, s_0)) < 0$. By Lemma 5.1 $J(v(\cdot, s)) < 0$ for all $s > s_0$. Let us consider the quantity

$$F(s) = \int_{\mathbb{S}^n} v^{N+1}(y, s) dy \geq 0, \quad s \in (0, \infty).$$

Then

$$\begin{aligned} \frac{N}{N+1} \frac{d}{ds} F(s) &= \int_{\mathbb{S}^n} (v^N)_s v = -2J(v(\cdot, s)) + \frac{N-1}{(1-m)(N+1)} F(s) \\ &\geq \frac{N-1}{(1-m)(N+1)} F(s) \end{aligned}$$

for all $s > s_0$. Note that $F(s) \neq 0$ for all $s \geq s_0$, since otherwise $v(\cdot, s) \equiv 0$. But this is impossible because $J(v(\cdot, s)) \leq J(v(\cdot, s_0)) < 0$. Integrating the above differential inequality, we have

$$F(s) \geq F(s_0) e^{s-s_0}.$$

It follows that $F(s) \rightarrow \infty$ as $s \rightarrow \infty$. On the other hand, Proposition 5.1 tells us that v is uniformly bounded. Hence $F(s)$ is bounded. We reach a contradiction. \square

Proof of Theorem 5.1. It follows from Proposition 5.1 and Proposition 4.4 that for $s > s^*/2$, $v(\cdot, s)$ is compact in $C^k(\mathbb{S}^n)$ for any k . Let \bar{v} be a limit point of $v(\cdot, s)$ as $s \rightarrow \infty$ in the C^2 sense. We will show that \bar{v} is a solution of (50) and \bar{v} is the unique limit of $v(\cdot, s)$ as $s \rightarrow \infty$.

Suppose that along a sequence $s_j \rightarrow \infty$, $v(\cdot, s_j) \rightarrow \bar{v}$ in the C^2 sense. Since

$$\frac{d}{ds} J(v(\cdot, s)) = -N \int_{\mathbb{S}^n} v^{N-1} v_s^2 = -\frac{4N}{(N+1)^2} \int_{\mathbb{S}^n} |(v^{(N+1)/2}(\cdot, s))_s|,$$

we have, by integrating from s_j to $s_j + \tau$ and using the Cauchy-Schwarz inequality,

$$\begin{aligned} &\int_{\mathbb{S}^n} |v^{\frac{N+1}{2}}(\cdot, s_j + \tau) - v^{\frac{N+1}{2}}(\cdot, s_j)|^2 \\ &\leq \frac{(N+1)^2 \tau}{4N} (J(v(\cdot, s_j)) - J(v(\cdot, s_j + \tau))). \end{aligned}$$

By Lemma 5.1 and Lemma 5.2, $J(v(\cdot, s))$ has a limit as $s \rightarrow \infty$. Hence for each $\tau > 0$, $\{v(\cdot, s_j + \tau)\}_1^\infty$ is Cauchy in L^{N+1} . It follows that $v(\cdot, s_j + \tau) \rightarrow \bar{v}$ in L^{N+1} , and in C^2 sense uniformly in τ in bounded intervals. Thus for any $\phi \in C^\infty(\mathbb{S}^n)$ we have,

$$\begin{aligned} &\int_{\mathbb{S}^n} (v^N(\cdot, s_j + 1) - v^N(\cdot, s_j)) \phi \\ &= \int_0^1 \int_{\mathbb{S}^n} \left(-P_\sigma(v(y, s_j + \tau)) + \frac{1}{1-m} v^N(y, s_j + \tau) \right) \phi dy d\tau. \end{aligned}$$

After sending $j \rightarrow \infty$, we obtain

$$\int_{\mathbb{S}^n} \left(-P_\sigma(\bar{v}) + \frac{1}{1-m} \bar{v}^N \right) \phi = 0.$$

Hence \bar{v} solves (50). Finally, it follows from Theorem A.2 that $v(\cdot, s)$ converges to \bar{v} in $C^3(\mathbb{S}^n)$. \square

Proof of Theorem 1.2. By the classification of solutions of (50) in [22] and [54], Theorem 1.2 follows from Theorem 5.1 immediately. \square

From Theorem 1.2 we see that the extinction profile of $u(x, t)$ is determined by the pair of numbers $(\lambda, x_0) = (\lambda(u_0), x_0(u_0))$. The next theorem verifies the stability of both the extinction time and the extinction profile.

Theorem 5.2. *$T(u_0), \lambda(u_0)$ and $x_0(u_0)$ continuously depend on u_0 in the sense that if $u_0, \{u_{0;j}\}$ are positive C^2 functions in \mathbb{R}^n , $(u_0^m)_{0,1}, (u_{0;j}^m)_{0,1}$ can be extended to positive C^2 functions near the origin, and $\lim_{j \rightarrow \infty} \|u_{0;j}^m - u_0^m\|_b = 0$ where $\|\cdot\|_b$ is defined by*

$$\|\cdot\|_b = \|\cdot\|_{C^2(B_2)} + \|(\cdot)_{0,1}\|_{C^2(B_2)},$$

then

$$\lim_{j \rightarrow \infty} (T(u_{0;j}), \lambda(u_{0;j}), x_0(u_{0;j})) = (T(u_0), \lambda(u_0), x_0(u_0)).$$

Proof. Since we have Theorem A.1, Lemma A.1 and Theorem A.2, it is identical to the proof of Theorem 1.2 in [29]. We refer to [29] for details. \square

6 A Sobolev inequality and a Hardy-Littlewood-Sobolev inequality along a fractional diffusion equation

As mentioned in the introduction, the results in this section are inspired by [20] and [31].

Proposition 6.1. *Assume that $n \geq 2$. If u is a solution of (11) with positive initial data $u_0 \in C^2$ in \mathbb{R}^n satisfying that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin, then*

$$\frac{1}{2} \frac{d}{dt} H = \left(\int_{\mathbb{R}^n} u^{m+1} \right)^{\frac{2\sigma}{n}} \left(S_{n,\sigma} \|u^m\|_{\dot{H}^\sigma}^2 - \|u^m\|_{L^{2^*(\sigma)}}^2 \right) \geq 0,$$

where H is given by (17).

Proof. It follows from (11) and (13) that

$$\begin{aligned}
\frac{d}{dt}H &= \int_{\mathbb{R}^n} 2u(-\Delta)^{-\sigma} u_t dx - 2S_{n,\sigma} \left(\int_{\mathbb{R}^n} u^{m+1} \right)^{\frac{2\sigma}{n}} \int_{\mathbb{R}^n} u^m u_t \\
&= -2 \int_{\mathbb{R}^n} u^{m+1} + 2S_{n,\sigma} \left(\int_{\mathbb{R}^n} u^{m+1} \right)^{\frac{2\sigma}{n}} \int_{\mathbb{R}^n} u^m (-\Delta)^{\sigma} u^m \\
&= 2 \left(\int_{\mathbb{R}^n} u^{m+1} \right)^{\frac{2\sigma}{n}} \left(S_{n,\sigma} \|u^m\|_{\dot{H}^{\sigma}}^2 - \|u^m\|_{L^{2^*(\sigma)}}^2 \right) \geq 0.
\end{aligned}$$

Note that the first part of Theorem 5.1, i.e., v defined by (48) is positive and smooth in $\mathbb{S}^n \times (0, \infty)$, has been used in the justifications of these equalities. \square

The next lemma gives an estimate for the extinction time of solutions of (11).

Lemma 6.1. *If u is a solution of (11) with positive initial data $u_0 \in C^2$ in \mathbb{R}^n satisfying that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin, then for any $t \in (0, T)$ we have*

$$\left(\frac{4\sigma(T-t)}{(n+2\sigma)S_{n,\sigma}} \right)^{\frac{n}{2\sigma}} \leq \int_{\mathbb{R}^n} u^{m+1}(t, x) dx \leq \int_{\mathbb{R}^n} u_0^{m+1} dx.$$

Consequently, the extinction time T is bounded by

$$T \leq \frac{(n+2\sigma)S_{n,\sigma}}{4\sigma} \left(\int_{\mathbb{R}^n} u_0^{m+1} dx \right)^{\frac{2\sigma}{n}}.$$

If in addition $n > 4\sigma$, then

$$T \geq \frac{(n+2\sigma)}{2n} \frac{\int_{\mathbb{R}^n} u_0^{m+1} dx}{\int_{\mathbb{R}^n} u_0^m (-\Delta)^{\sigma} u_0^m}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} u^m(\cdot, t) (-\Delta)^{\sigma} u^m(\cdot, t) &\leq \int_{\mathbb{R}^n} u_0^m (-\Delta)^{\sigma} u_0^m \\
\int_{\mathbb{R}^n} u^{m+1}(\cdot, t) &\geq \int_{\mathbb{R}^n} u_0^{m+1} - \frac{2n}{n+2\sigma} t \int_{\mathbb{R}^n} u_0^m (-\Delta)^{\sigma} u_0^m.
\end{aligned}$$

Proof. As in the proof of Lemma 5.2, we define

$$F(t) := \int_{\mathbb{R}^n} u^{m+1}(x, t) dx, \tag{52}$$

which is positive in $(0, T)$ and $F(T) = 0$. It follows that

$$\begin{aligned} F'(t) &= (m+1) \int_{\mathbb{R}^n} u^m(\cdot, t) u_t(\cdot, t) \\ &= -(m+1) \int_{\mathbb{R}^n} u^m(\cdot, t) (-\Delta)^\sigma u^m(\cdot, t) \leq -\frac{m+1}{S_{n,\sigma}} F(t)^{1-\frac{2\sigma}{n}}, \end{aligned}$$

where we have used the Sobolev inequality (13) in the last inequality. This shows the first two inequalities by simple integrations. If in addition $n > 4\sigma$, then

$$\begin{aligned} F''(t) &= m(m+1) \int_{\mathbb{R}^n} u^{m-1}(\cdot, t) ((-\Delta)^\sigma u^m(\cdot, t))^2 \\ &\quad + m(m+1) \int_{\mathbb{R}^n} u^m(\cdot, t) (-\Delta)^\sigma (u^{m-1} (-\Delta)^\sigma u^m(\cdot, t)) \\ &= 2m(m+1) \int_{\mathbb{R}^n} u^{m-1}(\cdot, t) ((-\Delta)^\sigma u^m(\cdot, t))^2 \geq 0, \end{aligned}$$

where the condition $n > 4\sigma$ is used to guarantee the L^2 integrability of $u^m(\cdot, t)$ such that we can use Plancherel's theorem in the second equality. Thus the lower bound of T follows from that $0 = F(T) \geq F(t) + F'(t)(T-t)$ with sending $t \rightarrow 0$. The last two inequalities follows from the sign of F'' and simple integrations. \square

Let

$$Q := -\frac{1}{m+1} F' F^{\frac{2\sigma-n}{n}}, E := -\frac{1}{m+1} F' F^{-1}, G(t_1, t_2) := \exp\left((m+1) \int_{t_1}^{t_2} E(s) ds\right). \quad (53)$$

Theorem 6.1. Assume $n > 4\sigma$. For any u_0 positive and C^2 in \mathbb{R}^n satisfying that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin, we have

$$\begin{aligned} S_{n,\sigma} \|u_0\|_{L^{\frac{2n}{n+2\sigma}}}^2 - \int_{\mathbb{R}^n} u_0 (-\Delta)^{-\sigma} u_0 dx + 4m S_{n,\sigma} \int_0^T dt \int_0^t F(s)^{\frac{2\sigma}{n}} K(s) G(t, s) ds \\ = 2 \|u_0^m\|_{L^{2^*(\sigma)}^{\frac{4\sigma}{n-2\sigma}}}^2 \left(S_{n,\sigma} \|u_0^m\|_{\dot{H}^\sigma}^2 - \|u_0^m\|_{L^{2^*(\sigma)}}^2 \right) \int_0^T G(t, 0) dt \end{aligned}$$

where $u(\cdot, t)$ is the solution of (11) with initial data $u(\cdot, 0) = u_0$, T is the extinction time of $u(\cdot, t)$ and F, E, G, K are defined in (52), (53) and (54).

Proof. From the proof of Proposition 6.1 we know that

$$H'(t) = 2F(t)(S_{n,\sigma} Q(t) - 1).$$

Hence

$$\begin{aligned}
H''(t) &= 2F'(t)(S_{n,\sigma}Q(t) - 1) + 2F(t)S_{n,\sigma}Q'(t) \\
&= \frac{F'(t)}{F(t)}H'(t) + 2F(t)S_{n,\sigma}Q'(t) \\
&= -(m+1)E(t)H'(t) + 2F(t)S_{n,\sigma}Q'(t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
Q'(t) &= \frac{F''(t) - \frac{n-2\sigma}{n}F^{-1}(t)(F'(t))^2}{-(m+1)F(t)^{\frac{n-2\sigma}{n}}} \\
&= -\frac{2m}{F(t)^{\frac{n-2\sigma}{n}}} \left(\int_{\mathbb{R}^n} u^{m-1}(\cdot, t) ((-\Delta)^\sigma u^m(\cdot, t))^2 - F^{-1} \int_{\mathbb{R}^n} u^m(\cdot, t) (-\Delta)^\sigma u^m(\cdot, t) \right) \\
&= -\frac{2m}{F(t)^{\frac{n-2\sigma}{n}}} \int_{\mathbb{R}^n} u(\cdot, t)^{m-1} |(-\Delta)^\sigma u(\cdot, t)^m + E(t)u(\cdot, t)|^2.
\end{aligned}$$

Denote

$$K(t) := \int_{\mathbb{R}^n} u(\cdot, t)^{m-1} |(-\Delta)^\sigma u(\cdot, t)^m + E(t)u(\cdot, t)|^2. \quad (54)$$

Then

$$H''(t) = -(m+1)E(t)H'(t) - 4mF^{\frac{2\sigma}{n}}(t)S_{n,\sigma}K(t).$$

Multiplying $G(0, s)$ and integrating from 0 to t , we have

$$H'(t)G(0, t) - H'(0)G(0, 0) = \int_0^t (H'G)'(s)ds = -4mS_{n,\sigma} \int_0^t F(s)^{\frac{2\sigma}{n}} K(s)G(0, s)ds.$$

Dividing $G(0, t)$ and integrating from 0 to T , we obtain

$$0 - H(0) = H'(0) \int_0^T G(t, 0)dt - 4mS_{n,\sigma} \int_0^T dt \int_0^t F(s)^{\frac{2\sigma}{n}} K(s)G(t, s)ds.$$

□

The drawback of the above Theorem is that the extra terms are not explicit. Fortunately we can use simple estimates to reach Theorem 1.3.

Proof of Theorem 1.3. We first assume that $w = u_0^m$ where $u_0 \in C^2(\mathbb{R}^n)$ is positive and satisfies that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin. By Lemma 6.1,

$$(m+1)E(s) \geq (m+1)S_{n,\sigma}^{-1} \left(\int_{\mathbb{R}^n} u(\cdot, s)^{m+1} \right)^{-2\sigma/n} \geq (m+1)S_{n,\sigma}^{-1} \left(\int_{\mathbb{R}^n} u_0^{m+1} \right)^{-2\sigma/n} =: b.$$

By Lemma 6.1 again, we have $bT \leq \frac{n}{2\sigma}$. Therefore,

$$\begin{aligned} \int_0^T G(t, 0) dt &\leq \int_0^T e^{-bt} dt = \frac{1 - e^{-bT}}{b} \\ &\leq \frac{1 - e^{-\frac{n}{2\sigma}}}{m+1} S_{n,\sigma} \left(\int_{\mathbb{R}^n} u_0^{m+1} \right)^{2\sigma/n}. \end{aligned}$$

Hence (18) holds for $w = u_0^m$ where $u_0 \in C^2(\mathbb{R}^n)$ is positive and satisfies that $(u_0^m)_{0,1}$ can be extended to a positive C^2 function near the origin.

For any nonnegative $u \in C_c^\infty(\mathbb{R}^n)$, we consider $w_\varepsilon = u + \varepsilon \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2\sigma}{2}}$ with $\varepsilon > 0$. Then (18) holds for w_ε . By sending $\varepsilon \rightarrow 0$, we have (18) for u . Finally, Theorem 1.3 follows from a density argument. \square

A Uniqueness theorem for negative gradient flow involving nonlocal operator

In this appendix, we provide a uniqueness theorem for fractional Yamabe flows, which is analog to L. Simon's uniqueness Theorem in [68]. The proofs are essentially the same and we will just sketch them in our setting. Denote $H^\sigma(\mathbb{S}^n)$ as the closure of $C^\infty(\mathbb{S}^n)$ under the norm

$$\|v\|_{H^\sigma(\mathbb{S}^n)} = \int_{\mathbb{S}^n} v P_\sigma(v).$$

Let $\alpha \in (0, 1)$ such that $2\sigma + \alpha$ is not an integer. Let J be the functional defined as

$$J(v) = \frac{1}{2} \int_{\mathbb{S}^n} v P_\sigma(v) - \frac{1}{(1-m)(N+1)} \int_{\mathbb{S}^n} v^{N+1}, \quad v \in H^\sigma(\mathbb{S}^n).$$

Then

$$\nabla J(v) = P_\sigma(v) - \frac{1}{1-m} v^N.$$

Let \bar{v} be such that $\nabla J(\bar{v}) = 0$.

Theorem A.1. *There exist $\theta \in (0, 1/2)$ and $r_0 > 0$ such that for any $v \in C^{2\sigma+\alpha}(\mathbb{S}^n)$ with $\|v - \bar{v}\|_{C^{2\sigma+\alpha}} < r_0$,*

$$\|\nabla J(v)\|_{L^2(\mathbb{S}^n)} \geq |J(v) - J(\bar{v})|^{1-\theta}.$$

Proof. Since we have Schauder estimates (see, e.g., [48]) and L^2 estimates (which is free from equivalence of definitions of fractional Sobolev spaces on \mathbb{S}^n) for P_σ , the proof is identical to that of Theorem 3 in [68]. \square

Let $v(x, s)$ and \bar{v} be as in Section 5. Then direction computations and uniform bounds of $v(x, s)$ yield the following lemma

Lemma A.1. *There exist two constant c_0 and T_0 such that for any $t > T_0$ we have*

$$-\frac{d}{ds}J(v(\cdot, s)) \geq c_0 \|v_s\|_{L^2(\mathbb{S}^n)} \|\nabla J(v(\cdot, s))\|_{L^2(\mathbb{S}^n)}.$$

Theorem A.2.

$$\lim_{s \rightarrow \infty} \|v(\cdot, s) - \bar{v}\|_{C^l(\mathbb{S}^n)} = 0$$

for any positive integer l .

Proof. First we can prove that $v(\cdot, t)$ converges to \bar{v} in $C^{2\sigma+\alpha}(\mathbb{S}^n)$, using the same methods as the proof of Proposition 21 in [1] or the proof of Theorem 1 in [40]. Then Theorem A.2 follows from the uniform C^{l+1} bound of $v(x, s)$. \square

Similarly if

$$S(z) = \frac{\int_{\mathbb{S}^n} z P_\sigma(z)}{\left(\int_{\mathbb{S}^n} z^{N+1}\right)^{\frac{2}{N+1}}}, \quad z \in H^\sigma(\mathbb{S}^n),$$

then

$$\nabla S(z) = 2 \left(\int_{\mathbb{S}^n} z^{N+1} \right)^{-\frac{2}{N+1}} \left(P_\sigma(z) - \frac{\int_{\mathbb{S}^n} z P_\sigma(z)}{\int_{\mathbb{S}^n} z^{N+1}} z^N \right).$$

Let $v(x, t)$ and v_∞ be as in Theorem 1.1. Note that $\nabla S(v_\infty) = 0$.

Theorem A.3. *There exist $\theta \in (0, 1/2)$ and $r_0 > 0$ such that for any $\|v - v_\infty\|_{C^{2\sigma+\alpha}} < r_0$,*

$$\|\nabla S(v)\|_{L^2(\mathbb{S}^n)} \geq |S(v) - S(v_\infty)|^{1-\theta}.$$

Lemma A.2. *There exist two constant c_0 and T_0 such that for any $t > T_0$ we have*

$$-\frac{d}{ds}S(v(\cdot, s)) \geq c_0 \|v_t\|_{L^2(\mathbb{S}^n)} \|\nabla S(v(\cdot, t))\|_{L^2(\mathbb{S}^n)}.$$

Theorem A.4.

$$\lim_{t \rightarrow \infty} \|v(\cdot, t) - v_\infty\|_{C^l(\mathbb{S}^n)} = 0$$

for any positive integer l .

B Some interpolation inequalities

Lemma B.1. *Suppose that $0 < \alpha < \min(1, 2\sigma)$ and $2\sigma + \alpha$ is not an integer. There exists a constant $C > 0$ depending only on n and σ such that for any $\varepsilon > 0$ and $u \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_T)$, we have*

$$|u_t|_{0;Q_T} \leq \varepsilon[u_t]_{\alpha, \frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-2\sigma/\alpha}|u|_{0;Q_T}, \quad (55)$$

$$|(-\Delta)^\sigma u|_{0;Q_T} \leq \varepsilon[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-2\sigma/\alpha}|u|_{0;Q_T}, \quad (56)$$

$$[u]_{\alpha, \frac{\alpha}{2\sigma};Q_T} \leq \varepsilon[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-\alpha/(2\sigma)}|u|_{0;Q_T}. \quad (57)$$

If $\sigma > \frac{1}{2}$, then

$$[\nabla_x u]_{\alpha, \frac{\alpha}{2\sigma};Q_T} \leq \varepsilon[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_T} + C\varepsilon^{-(1+\alpha)/(2\sigma-1)}|u|_{0;Q_T}. \quad (58)$$

Proof. By the fractional parabolic dilations of the form $u(x, t) \rightarrow u(Rx, R^{2\sigma}t)$, we only need to show the case $\varepsilon = 1$ and $T = 2$. Take $X = (x, t) \in Q_T$ and we have, for some $\theta \in (-1, 1)$,

$$\begin{aligned} |u_t(X)| &\leq |u_t(X) - (u(x, t \pm 1) - u(x, t))| + 2|u|_{0;Q_T} \\ &= |u_t(X) - u_t(x, t + \theta)| + 2|u|_{0;Q_T} \leq [u_t]_{\alpha, \frac{\alpha}{2\sigma};Q_T} + 2|u|_{0;Q_T}. \end{aligned}$$

This finishes the proof of (55). For (56) and (57), we first recall (see, e.g., [69]) that

$$|w|_{2\sigma+\alpha; \mathbb{R}^n} \leq C(|w|_{0; \mathbb{R}^n} + |(-\Delta)^\sigma w|_{\alpha; \mathbb{R}^n}) \quad \text{for all } w \in C^{2\sigma+\alpha}(\mathbb{R}^n).$$

Hence

$$\begin{aligned} |(-\Delta)^\sigma u(x, t)| &\leq C(|u(\cdot, t)|_{0; \mathbb{R}^n} + |u(\cdot, t)|_{C^{2\sigma+\alpha}(\mathbb{R}^n)}) \\ &\leq C(|u|_{0;Q_T} + [(-\Delta)^\sigma u]_{\alpha, \frac{\alpha}{2\sigma};Q_T}) \leq C([u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}), \end{aligned}$$

and

$$\begin{aligned} [u]_{\alpha, \frac{\alpha}{2\sigma};Q_T} &\leq \sup_{t_1 \neq t_2, x} \frac{|u(x, t_1) - u(x, t_2)|}{|t_1 - t_2|^{\frac{\alpha}{2\sigma}}} + \sup_{x_1 \neq x_2, t} \frac{|u(x_1, t) - u(x_2, t)|}{|x_1 - x_2|^\alpha} \\ &\leq C(|u|_{0;Q_T} + |u_t|_{0;Q_T} + \sup_t |u(\cdot, t)|_{2\sigma+\alpha; \mathbb{R}^n}) \\ &\leq C([u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}). \end{aligned}$$

Finally, for $\sigma > \frac{1}{2}$ we notice that by the same methods as above,

$$\sup_{t, x_1 \neq x_2} \frac{|\nabla_x u(x_1, t) - \nabla_x u(x_2, t)|}{|x_1 - x_2|^\alpha} \leq C([u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma};Q_T} + |u|_{0;Q_T}).$$

Hence to prove (58), we only need to show

$$\sup_{s \neq t, x} \frac{|\nabla_x u(x, s) - \nabla_x u(x, t)|}{|s - t|^{\frac{\alpha}{2\sigma}}} \leq C([u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} + |u|_{0; Q_T}).$$

Fix any $x_0 \in \mathbb{R}^n$. Let $w(x, t) = (-\Delta)^\sigma u(x, t)$ and $\eta(x)$ be a smooth cut-off function supported in $B_2(x_0) \in \mathbb{R}^n$ and equal to 1 in $B_1(x_0)$. Let

$$u_0(x, t) = (-\Delta)^{-\sigma}(\eta w) = \int_{\mathbb{R}^n} \frac{\eta(y)w(y, t)}{|x - y|^{n-2\sigma}} dy.$$

For convenience we have omitted some positive constant as in (16). Then

$$(-\Delta)^\sigma(u_0(x, t) - u(x, t) - u_0(x, s) + u(x, s)) = 0 \quad \text{in } B_1(x_0),$$

which implies, for $0 < |t - s| \leq 1$,

$$\begin{aligned} & |\nabla_x u_0(x_0, t) - \nabla_x u(x_0, t) - \nabla_x u_0(x_0, s) + \nabla_x u(x_0, s)| \\ & \leq C|u_0(x, t) - u(x, t) - u_0(x, s) + u(x, s)|_{L^\infty(\mathbb{R}^n)} \\ & \leq C(|u_t|_{0; Q_T} + [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T})|t - s|^{\frac{\alpha}{2\sigma}}. \end{aligned}$$

Since $\sigma > 1/2$ and

$$\nabla_x u_0(x_0, t) = (2\sigma - n) \int_{\mathbb{R}^n} \frac{(x_0 - y)\eta(y)w(y, t)}{|x_0 - y|^{n+2-2\sigma}} dy,$$

we have

$$|\nabla_x u_0(x_0, t) - \nabla_x u_0(x_0, s)| \leq C[u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} |t - s|^{\frac{\alpha}{2\sigma}}.$$

Thus together with (55)

$$\sup_{s \neq t, x} \frac{|\nabla_x u(x, s) - \nabla_x u(x, t)|}{|s - t|^{\frac{\alpha}{2\sigma}}} \leq C([u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} + |u|_{0; Q_T}).$$

This finishes the proof of (58). \square

Lemma B.2. Suppose that $0 < \alpha < \min(1, 2\sigma)$ and $2\sigma + \alpha$ is not an integer. For any small $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ depending only on n, σ and ε such that for any $v \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_T)$, we have

$$|v_t|_{0; Q_T} \leq \varepsilon[v]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} + C(\varepsilon)|v|_{0; Q_T}, \quad (59)$$

$$|P_\sigma v|_{0; Q_T} \leq \varepsilon[v]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} + C(\varepsilon)|v|_{0; Q_T}, \quad (60)$$

$$[v]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} \leq \varepsilon[v]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} + C(\varepsilon)|v|_{0; Q_T}. \quad (61)$$

Proof. Using stereographic projections, (9) and noticing that $|x - y| \geq C_n |F(x) - F(y)|$, the above inequalities follows from Lemma B.1. \square

Lemma B.3. *Suppose that $0 < \alpha < \min(1, 2\sigma)$ and $2\sigma + \alpha$ is not an integer. Let $u \in \mathcal{C}^{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}}(Q_1)$ and $\eta \in C_c^2(\mathbb{R}^{n+1})$, then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ depending only on $\alpha, \sigma, n, \varepsilon$ and $\|\eta\|_{C^2(\mathbb{R}^{n+1})}$ such that*

$$[\langle u, \eta \rangle]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \leq \varepsilon [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} + C(\varepsilon) |u|_{0, Q_1}. \quad (62)$$

Proof. We denote C as various constants depending only on $n, \sigma, \alpha, \|\eta\|_{C^2(\mathbb{R}^{n+1})}$, and $C(\varepsilon)$ as various constants depending only on $n, \sigma, \alpha, \|\eta\|_{C^2(\mathbb{R}^{n+1})}$ and ε . Recall that $\langle u, \eta \rangle$ is defined in (35). For any $(x, t) \in Q_1$,

$$\begin{aligned} |\langle u, \eta \rangle(x, t)| &\leq c(n, \sigma) \int_{\mathbb{R}^n \setminus B_1(x)} \frac{|u(x, t) - u(y, t)| |\eta(x, t) - \eta(y, t)|}{|x - y|^{n+2\sigma}} dy \\ &\quad + c(n, \sigma) \int_{B_1(x)} \frac{|u(x, t) - u(y, t)| |\eta(x, t) - \eta(y, t)|}{|x - y|^{n+2\sigma}} dy \\ &\leq C |u|_{0, Q_1} + C [u(\cdot, t)]_{\sigma, \mathbb{R}^n} \leq \varepsilon [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_1} + C(\varepsilon) |u|_{0, Q_1}. \end{aligned}$$

Fix any $X_1 = (x_1, t_1), X_2 = (x_2, t_2) \in Q_1, X_1 \neq X_2$. For convenience, we write $\rho = \rho(X_1, X_2)$ and $u^z(x, t) = u(x, t) - u(x + z, t)$. We may also suppose that $\rho \leq 1$.

$$\begin{aligned} &|\langle u, \eta \rangle(x_1, t_1) - \langle u, \eta \rangle(x_2, t_2)| \\ &\leq \left| \int_{|z| \leq \rho} \frac{(u^z(x_1, t_1) - u^z(x_2, t_2)) \eta^z(x_1, t_1)}{|z|^{n+2\sigma}} dz \right| \\ &\quad + \left| \int_{|z| \leq \rho} \frac{(\eta^z(x_1, t_1) - \eta^z(x_2, t_2)) u^z(x_2, t_2)}{|z|^{n+2\sigma}} dz \right| \\ &\quad + \left| \int_{|z| \geq \rho} \frac{(u^z(x_1, t_1) - u^z(x_2, t_2)) \eta^z(x_1, t_1)}{|z|^{n+2\sigma}} dz \right| \\ &\quad + \left| \int_{|z| \geq \rho} \frac{(\eta^z(x_1, t_1) - \eta^z(x_2, t_2)) u^z(x_2, t_2)}{|z|^{n+2\sigma}} dz \right| \\ &:= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

For I_1 and I_2 , we first consider that $2\sigma + \alpha < 1$. Then by change of variable,

$$I_1 + I_2 \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+\sigma; \mathbb{R}^n} \int_{|z| \leq \rho} |z|^{\alpha+\sigma+1-n-2\sigma} dz \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+\sigma; \mathbb{R}^n} \rho^{1+\alpha-\sigma}.$$

If $1 < \alpha + 2\sigma < 2$, we have

$$I_1 + I_2 \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+2\sigma-1; \mathbb{R}^n} \int_{|z| \leq \rho} |z|^{\alpha+2\sigma-n-2\sigma} dz \leq C \max_{i=1,2} [u(\cdot, t_i)]_{\alpha+2\sigma-1; \mathbb{R}^n} \rho^\alpha.$$

If $2\sigma + \alpha > 2$, then

$$\begin{aligned} I_1 &\leq \left| \int_{|z| \leq \rho} \frac{(u^z(x_1, t_1) + \nabla_x u(x_1, t_1)z - u^z(x_2, t_2) - \nabla_x u(x_2, t_2)z) \eta^z(x_1, t_1)}{|z|^{n+2\sigma}} dz \right| \\ &\quad + \left| \int_{|z| \leq \rho} \frac{(\nabla_x u(x_1, t_1)z - \nabla_x u(x_2, t_2)z) \eta^z(x_1, t_1)}{|z|^{n+2\sigma}} dz \right| \\ &\leq C \sup_{Q_1} |\nabla_x^2 u| \int_{|z| \leq \rho} |z|^{3-n-2\sigma} dz + C [\nabla_x u]_{\alpha, \frac{\alpha}{2\sigma}; Q_T} \rho^\alpha \int_{|z| \leq \rho} |z|^{2-n-2\sigma} dz \\ &\leq \rho^\alpha (\varepsilon [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} + C(\varepsilon) |u|_{0; Q_T}). \end{aligned}$$

Similarly

$$\begin{aligned} I_2 &\leq C |\nabla_x u|_{0; Q_1} \int_{|z| \leq \rho} |z|^{3-n-2\sigma} dz + C |\nabla_x u|_{0; Q_1} \rho^\alpha \int_{|z| \leq \rho} |z|^{2-n-2\sigma} dz \\ &\leq \rho^\alpha (\varepsilon [u]_{2\sigma+\alpha, 1+\frac{\alpha}{2\sigma}; Q_T} + C(\varepsilon) |u|_{0; Q_T}). \end{aligned}$$

For I_3 and I_4 we first consider that $\sigma \leq \frac{1}{2}$. Choose an $\alpha' > \alpha$ but sufficiently close to α such that $\alpha' < \min(1, 2\sigma)$, then

$$\begin{aligned} I_3 &\leq [u]_{\alpha', \frac{\alpha'}{2\sigma}; Q_1} \rho^{\alpha'} C \int_{|z| \geq \rho} |z|^{2\sigma+\alpha-\alpha'-n-2\sigma} dz \leq C [u]_{\alpha', \frac{\alpha'}{2\sigma}; Q_1} \rho^\alpha, \\ I_4 &\leq C \rho^{\alpha'} [u(\cdot, t_2)]_{2\sigma+\alpha-\alpha'; \mathbb{R}^n} \int_{|z| \geq \rho} |z|^{2\sigma+\alpha-\alpha'-n-2\sigma} dz \leq C [u(\cdot, t_2)]_{2\sigma+\alpha-\alpha'; \mathbb{R}^n} \rho^\alpha. \end{aligned}$$

If $\sigma > \frac{1}{2}$ and $2\sigma + \alpha < 2$, then

$$\begin{aligned} I_3 &\leq [u]_{2\sigma+\alpha-1, \frac{2\sigma+\alpha-1}{2\sigma}; Q_1} \rho^{2\sigma+\alpha-1} C \int_{|z| \geq \rho} |z|^{1-n-2\sigma} dz \leq C [u]_{2\sigma+\alpha-1, \frac{2\sigma+\alpha-1}{2\sigma}; Q_1} \rho^\alpha, \\ I_4 &\leq C \rho^{2\sigma+\alpha-1} |\nabla u(\cdot, t_2)|_{0; \mathbb{R}^n} \int_{|z| \geq \rho} |z|^{1-n-2\sigma} dz \leq C |\nabla u(\cdot, t_2)|_{0; \mathbb{R}^n} \rho^\alpha. \end{aligned}$$

If $2\sigma + \alpha > 2$, then for $\rho \leq |z| \leq 1$, we have

$$|u^z(x_1, t_1) - u^z(x_2)| \leq |\nabla_x^2 u|_{0; Q_1} |x_1 - x_2| |z| + |u_t|_{0; Q_1} |t_1 - t_2| \leq |\nabla_x^2 u|_{0; Q_1} \rho |z| + |u_t|_{0; Q_1} \rho^{2\sigma}.$$

Hence

$$\begin{aligned}
I_3 &\leq \left| \int_{1 \geq |z| \geq \rho} \frac{(u^z(x_1, t_1) - u^z(x_2, t_2)) \eta^z(x_1, t_1)}{|z|^{n+2\sigma}} dz \right| \\
&\quad + \left| \int_{|z| \geq 1} \frac{(u^z(x_1, t_1) - u^z(x_2, t_2)) \eta^z(x_1, t_1)}{|z|^{n+2\sigma}} dz \right| \\
&\leq C |\nabla_x^2 u|_{0; Q_1} \rho \int_{1 \geq |z| \geq \rho} |z|^{2-n-2\sigma} dz + C |u_t|_{0; Q_1} \rho^{2\sigma} \int_{1 \geq |z| \geq \rho} |z|^{1-n-2\sigma} \\
&\quad + [u]_{\alpha, \frac{\alpha}{2\sigma}; Q_1} \rho^\alpha \int_{|z| \geq 1} |z|^{-n-2\sigma} dz \\
&\leq C (|\nabla_x^2 u|_{0; Q_1} + |u_t|_{0; Q_1} + [u]_{\alpha, \frac{\alpha}{2\sigma}; Q_1}) \rho^\alpha
\end{aligned}$$

Similarly for I_4 we have

$$I_4 \leq C |\nabla_x u|_{0; Q_1} \rho^\alpha.$$

Combining these and applying some interpolation inequalities in Lemma B.1, we reach (62). \square

References

- [1] Andrews, B.: *Monotone quantities and unique limits for evolving convex hypersurfaces*, Internat. Math. Res. Notices **1997**, no. 20, 1001–1031
- [2] Athanassopoulos, I.; Caffarelli, L. A.: *Continuity of the temperature in boundary heat control problems*, Adv. Math., **224** (2010), 293–315
- [3] Aubin, T.: *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geom. **11** (1976), no. 4, 573–598
- [4] Baird, P.; Fardoun, A.; Regbaoui, R.: *Q-curvature flow on 4-manifolds*, Calc. Var. Partial Differential Equations **27** (2006), no. 1, 75–104
- [5] Beckner, W.: *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. **138** (1993), 213–242
- [6] Bianchi, G.; Egnell, H.: *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), no. 1, 18–24
- [7] Blanchet, A.; Bonforte, M.; Dolbeault, J.; Grillo, G.; Vázquez, J.L.: *Asymptotics of the fast diffusion equation via entropy estimates*, Arch. Ration. Mech. Anal. **191** (2009), no. 2, 347–385

- [8] Bonforte, M.; Dolbeault, J.; Grillo, G.; Vázquez, J. L.: *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proc. Natl. Acad. Sci. USA **107** (2010), no. 38, 16459–16464
- [9] Branson, T.P.: *Sharp inequalities, the functional determinant, and the complementary series*, Trans. Amer. Math. Soc., **347** (1995), 367–3742
- [10] Brendle, S.: *A generalization of the Yamabe flow for manifolds with boundary*, Asian J. Math. **6** (2002), no. 4, 625–644.
- [11] Brendle, S.: *Global existence and convergence for a higher order flow in conformal geometry*. Ann. of Math. (2) **158** (2003), no. 1, 323–343
- [12] Brendle, S.: *Convergence of the Yamabe flow for arbitrary initial energy*, J. Differential Geom. **69** (2005), 217–278
- [13] Brendle, S.: *Convergence of the Q -curvature flow on \mathbb{S}^4* , Adv. Math. **205** (2006), no. 1, 1–32
- [14] Brendle, S.: *Convergence of the Yamabe flow in dimension 6 and higher*, Invent. Math. **170** (2007), no. 3, 541–576
- [15] Brezis, H.; Lieb, E.: *Sobolev inequalities with remainder terms*, J. Funct. Anal. **62** (1985), no. 1, 73–86
- [16] Brézis, H.; Nirenberg, L.: *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.
- [17] Cabre, X.; Sire Y.: *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates* preprint, [arXiv:1012.0867](https://arxiv.org/abs/1012.0867)
- [18] Caffarelli, L.; Silvestre, L.: *An extension problem related to the fractional Laplacian*, Comm. Partial. Diff. Equ., **32** (2007), 1245–1260
- [19] Caffarelli, L.; Vázquez, J.L.: *Nonlinear porous medium flow with fractional potential pressure*, to appear in Arch. Rational Mech. Anal
- [20] Carlen, E.; Carrillo, J.; Loss, M.: *Hardy-Littlewood-Sobolev inequalities via fast diffusion flows*, Proc. Natl. Acad. Sci. USA, **107** (2010), no. 46, 19696–19701
- [21] Chang, S.-Y.; González, M.: *Fractional Laplacian in conformal geometry*, Adv. Math., **226** (2011), 1410–1432
- [22] Chen, W.; Li, C.; Ou, B.: *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., **59** (2006), 330–343

- [23] Cherrier, P.: *Problèmes de Neumann non linéaires sur les variétés riemanniennes*, J. Funct. Anal. **57** (1984), no. 2, 154–206
- [24] Chow, B.: *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math., **45** (1992), 1003–1014
- [25] Cianchi, A.; Fusco, N.; Maggi, F.; Pratelli, A.: *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 5, 1105–1139
- [26] Daskalopoulos, P.; Sesum, N.: *On the extinction profile of solutions to fast diffusion*, J. Reine Angew. Math. **622** (2008), 95–119
- [27] de Pablo, A.; Quiros, F.; Rodriguez, A.; Vazquez, J.: *A fractional porous medium equation*, Adv. Math. **226** (2011), no. 2, 1378–1409
- [28] de Pablo, A.; Quiros, F.; Rodriguez, A.; Vazquez, J.: *A general fractional porous medium equation*, arXiv:1104.0306v1
- [29] del Pino, M.; Sáez, M.: *On the extinction profile for solutions of $u_t = \Delta u^{(n-2)/(n+2)}$* , Indiana Univ. Math. J., **50** (2001), no. 1, 611–628
- [30] Di Nezza E.; Palatucci, G.; Valdinoci, E.: *Hitchhiker’s guide to the fractional Sobolev spaces*, arXiv:1104.4345v2
- [31] Dolbeault, J.: *Sobolev and Hardy-Littlewood-Sobolev inequalities: duality and fast diffusion*, arXiv:1103.1145v1
- [32] Escobar, J.: *The Yamabe problem on manifolds with boundary*, J. Differential Geom. **35** (1992), no. 1, 21–84
- [33] Escobar, J.: *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. (2) **136** (1992), no. 1, 1–50
- [34] Galaktionov, V.; Peletier, L.: *Asymptotic behaviour near finite-time extinction for the fast diffusion equation*, Arch. Rational Mech. Anal. **139** (1997), no. 1, 83–98
- [35] Gidas, B.; Ni, W.-M.; Nirenberg, L.: *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., **68** (1979), 209–243
- [36] González, M.; Mazzeo, R.; Sire, Y.: *Singular Solutions of Fractional Order Conformal Laplacians*, to appear in J. Geom. Anal.
- [37] Gonzalez, M.; Qing, J.: *Fractional conformal Laplacians and fractional Yamabe problems*, arXiv:1012.0579v1

- [38] Graham, R.; Jenne, R.; Mason, L.; Sparling, G.: *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc. (2) **46** (1992), no. 3, 557–565
- [39] Graham, C.R.; Zworski, M.: *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89–118
- [40] Guan, P.; Wang, G.: *A fully nonlinear conformal flow on locally conformally flat manifolds*, J. Reine Angew. Math. **557** (2003), 219–238
- [41] Hamilton, R.: *The Ricci flow on surfaces*, pp. 237–262 in : *Mathematics and General Relativity* J. Isenberg, ed. Contemp. Math. Vol. 71, AMS, 1988
- [42] Han, Z.-C.; Li, Y.Y.: *The Yamabe problem on manifolds with boundary: existence and compactness results*, Duke Math. J. **99** (1999), no. 3, 489–542
- [43] Han, Z.-C.; Li, Y.Y.: *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature*, Comm. Anal. Geom. **8** (2000), no. 4, 809–869
- [44] Ho, P. T.: *Q-curvature flow on \mathbb{S}^n* , Comm. Anal. Geom. **18** (2010), no. 4, 791–820
- [45] Jara, M.: *Hydrodynamic limit of particle systems with long jumps*, arXiv:0805.1326v2
- [46] Jara, M.: *Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps*, Comm. Pure Appl. Math. **62** (2009), no. 2, 198–214
- [47] Jara, M.; Komorowski, T.; Olla, S.: *Limit theorems for additive functionals of a Markov chain*, Ann. Appl. Probab. **19** (6) (2009) 2270–2300
- [48] Jin, T.; Li, Y.Y.; Xiong, J.: *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, preprint, 2011.
- [49] Jin, T.; Li, Y.Y.; Xiong, J.: *On a fractional Nirenberg problem, part II: existence of solutions*, in preparation, 2011.
- [50] Kim, S.; Lee, K-A.: *Hölder Estimates for Singular Non-local Parabolic Equations*, arXiv:1105.3286v1
- [51] Kassmann, M.: *The classical Harnack inequality fails for nonlocal operators*, submitted, available as SFB 611-preprint No. 360
- [52] Kochubei, A.: *Parabolic pseudodifferential equations, hypersingular integrals and Markov processes* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 5, 909–934, 1118; translation in Math. USSR-Izv. **33** (1989), no. 2, 233–259
- [53] Krylov, N.V.: *“Lectures on elliptic and parabolic equations in Hölder space,”* Graduate Studies in Mathematics, 12. American Mathematical Society, Providence, RI, 1996

- [54] Li, Y.Y.: *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc. **6** (2004), 153–180
- [55] Lieb, E.H.: *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. Math., **118** (1983), 349–374
- [56] Madych, W.R.; Rivi re, N.M.: *Multipliers of the H lder classes*, J. Funct. Anal., **21** (1976), 369–379
- [57] Malchiodi, A.; Struwe, M.: *Q-curvature flow on \mathbb{S}^4* , J. Differential Geom. **73** (2006), no. 1, 1–44
- [58] Mikulevi cius, R.; Pragarauskas, H.: *On the Cauchy problem for certain integro-differential operators in Sobolev and H lder spaces*. (Lithuanian summary) Liet. Mat. Rink. **32** (1992), no. 2, 299–331; translation in Lithuanian Math. J. **32** (1992), no. 2, 238–264
- [59] Mikulevi cius, R.; Pragarauskas, H.: *On H lder solutions of the integro-differential Zakai equation*, Stochastic Process. Appl. **119** (2009), no. 10, 3319–3355
- [60] Mikulevi cius, R.; Pragarauskas, H.: *On the Cauchy problem for integro-differential operators in H lder classes and the uniqueness of the martingale problem*, arXiv:1103.3492v1
- [61] Morpurgo, C.: *Sharp inequalities for functional integrals and traces of conformally invariant*, Duke Math. J. **114** (2002), 477–553
- [62] Nirenberg, L.: *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math. **6** (1953), 167–177
- [63] Pavlov, P.; Samko, S.: *A description of spaces $L_p^\alpha(S_{n-1})$ in terms of spherical hypersingular integrals* (Russian), Dokl. Akad. Nauk SSSR **276** (1984), no. 3, 546–550. English translation: Soviet Math. Dokl. **29** (1984), no. 3, 549–553
- [64] Qing, J.; Raske, D.: *On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds*, Int. Math. Res. Not. 2006, Art. ID 94172, 20 pp
- [65] Rubin, B.: *The inversion of fractional integrals on a sphere*, Israel J. Math. **79** (1992), no. 1, 47–81.
- [66] Schwetlick, H.; Struwe, M.: *Convergence of the Yamabe flow for “large” energies*, J. reine angew. Math. **562** (2003), 59–100
- [67] Silvestre, L.: *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., **60** (2007), 67–112.

- [68] Simon, L.: *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571
- [69] Stein, E.: “*Singular integrals and differentiability properties of function*”, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970
- [70] Talenti, G.: *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976) 353–372
- [71] Tan, J.; Xiong, J.: *A Harnack inequality for fractional Laplace equations with lower order terms*, Discrete Contin. Dyn. Syst. **31** (2011), no. 3, 975–983
- [72] Ye, R.: *Global existence and convergence of Yamabe flow*, J. Differential Geom. **39** (1994), 35–50

Tianling Jin

Department of Mathematics, Rutgers University

110 Frelinghuysen Road, Piscataway, NJ 08854, USA

Email: kingbull@math.rutgers.edu

Jingang Xiong

School of Mathematical Sciences, Beijing Normal University

Beijing 100875, China

and

Department of Mathematics, Rutgers University

110 Frelinghuysen Road, Piscataway, NJ 08854, USA

Email: jxiong@mail.bnu.edu.cn/jxiong@math.rutgers.edu